WHOLE-BODY BACKSTEPPING CONTROL OF A QUADROTOR UAV FOR TRAJECTORY TRACKING

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2018
To Luigi.
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To the members of MACRO and ProVant groups, and in special to colleagues Adriano, Davidson, Henrique, Michel, Daniel, Edson, Brenner, Iuro, Arthur and Fabiano.
Este trabalho propõe contribuições no desenvolvimento de estratégias de controle não linear para veículos aéreos não tripulados (VANT). Controles não lineares recursivos são projetados para um VANT do tipo quadrotor desempenhar rastreamento de trajetória variante no tempo, assumindo sua dinâmica completa com seis graus de liberdade (6DOF). Em uma primeira estratégia, é projetado um controlador backstepping com o objetivo de obter um vetor dos torques e empuxo total para rastreamento da trajetória de translação, com estabilização dos ângulos de rolagem e arfagem, sem haver o desacoplamento das dinâmicas de posição e orientação. Neste controle, singularidades no movimento de translação são evitadas através do uso da matriz de rotação como um estado. O controle do ângulo de guinada é desenvolvido separadamente através do mesmo procedimento backstepping. Ademais, é projetada ação integral dos erros das velocidades controladas, com o objetivo de garantir convergência para os sinais de referência na presença de incertezas paramétricas, dinâmicas não modeladas e distúrbios externos constantes. A prova de Estabilidade Assintótica Global Uniforme (UGAS) é feita através da teoria de Lyapunov e Teorema de Matrosov, incluindo a rejeição aos distúrbios. Na segunda estratégia de controle backstepping, em um primeiro passo são calculados o empuxo total para controle de altitude e a matriz de rotação desejada para o movimento de translação. Na sequência, o controlador gera os torques necessários para obtenção da matriz de rotação desejada, evitando pontos de singularidade nos movimentos de rotação e translação. A ação integral é novamente incluída no projeto do controlador para garantir convergência mesmo quando o sistema é submetido a ação de distúrbios. A prova de Estabilidade Assintótica Global Uniforme (UGAS) é feita para ambos os controladores através da Teoria de Lyapunov e Teorema de Matrosov. São realizadas simulações para corroborar a eficiência do controlador proposto ao rastrear trajetórias variantes no tempo das variáveis reguladas ($x$, $y$, $z$ e $\psi$).
Abstract

This thesis proposes contributions on nonlinear control strategies for unmaned aerial vehicles (UAV). Recursive nonlinear controls are designed to perform time-varying trajectory tracking of a quadrotor UAV, assuming its whole-body six degrees of freedom (6DOF) dynamics. In a first control strategy, a backstepping controller is designed in order to achieve a vector of torques and total thrust for the translational trajectory tracking and to stabilize the roll and pitch angles, without decoupling the position and orientation dynamics. In this controller, by using the rotation matrix as a state, singularities are avoided in the translational control. The yaw orientation control is carried out separately through the same backstepping procedure. Furthermore, integral action of the controlled velocities’ errors is included in the control design in order to guarantee convergence to the reference signals with the presence of parametric uncertainties, unmodeled dynamics and constant external disturbances. The second backstepping control strategy is developed, initially, computing the total thrust necessary to control the altitude and the desired rotation matrix to reach the translational motion. Secondly, the controller produces the necessary torques to achieve this desired rotation matrix, avoiding singularities in the rotational and translational movements. Again, the integral action is also combined in the control architecture ensuring convergence when the system is affected by disturbances. The Uniform Global Asymptotic Stability (UGAS) of both controllers are prove via Lyapunov theory and Matrosov’s Theorem. Simulations are carried out to corroborate the efficiency of the proposed controllers when required to track a time-varying trajectory of the regulated variables ($x$, $y$, $z$ and $\psi$).
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<tr>
<td>3D</td>
<td>Three-Dimensional</td>
</tr>
<tr>
<td>6DOF</td>
<td>Six Degrees of Freedom</td>
</tr>
<tr>
<td>CLF</td>
<td>Control Lyapunov Function</td>
</tr>
<tr>
<td>DOF</td>
<td>Degrees of Freedom</td>
</tr>
<tr>
<td>FF</td>
<td>Feed Forward</td>
</tr>
<tr>
<td>LQR</td>
<td>Linear Quadratic Regulator</td>
</tr>
<tr>
<td>MACRO</td>
<td>Mechatronics, Control, and Robotics</td>
</tr>
<tr>
<td>PD</td>
<td>Proportional-Derivative</td>
</tr>
<tr>
<td>PID</td>
<td>Proportional-Integral-Derivative</td>
</tr>
<tr>
<td>PID+FF</td>
<td>Proportional Integral Derivative with Feed Forward action</td>
</tr>
<tr>
<td>RPAS</td>
<td>Remotely-Piloted Aerial System</td>
</tr>
<tr>
<td>UAS</td>
<td>Unmanned Aerial System</td>
</tr>
<tr>
<td>UAV</td>
<td>Unmanned Aerial Vehicle</td>
</tr>
<tr>
<td>UFMG</td>
<td>Universidade Federal de Minas Gerais</td>
</tr>
<tr>
<td>UGAS</td>
<td>Uniform Global Asymptotically Stable</td>
</tr>
<tr>
<td>UGS</td>
<td>Uniform Global Stable</td>
</tr>
<tr>
<td>VTOL</td>
<td>Vertical Take-off and Landing</td>
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### Notation

<table>
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<th>Description</th>
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<tr>
<td>$O$</td>
<td>Zero matrix</td>
</tr>
<tr>
<td>$I$</td>
<td>Identity matrix</td>
</tr>
<tr>
<td>$SE(3)$</td>
<td>Special Euclidean group</td>
</tr>
<tr>
<td>$\mathcal{E}_x$</td>
<td>Error between a state $x$ and the reference trajectory $x_r$</td>
</tr>
<tr>
<td>$x_r$</td>
<td>Reference trajectory for a variable $x$</td>
</tr>
<tr>
<td>$x_0$</td>
<td>Initial condition of a variable $x$</td>
</tr>
<tr>
<td>$\dot{x}$</td>
<td>Time derivative of the variable $x$</td>
</tr>
<tr>
<td>$x^\top$</td>
<td>Transpose of the vector $x$</td>
</tr>
<tr>
<td>$A^\top$</td>
<td>Transpose of the matrix $A$</td>
</tr>
<tr>
<td>$A^{-1}$</td>
<td>Inverse of matrix $A$</td>
</tr>
<tr>
<td>$x \in \mathbb{R}^n$</td>
<td>Vector $x$ of $n$ dimension</td>
</tr>
<tr>
<td>$A \in \mathbb{R}^{n \times m}$</td>
<td>Matrix $A$ of $n \times m$ dimension</td>
</tr>
<tr>
<td>$\mathcal{X}_{z_i}$</td>
<td>Integral of the variable $z_i$ given by $\int_0^t z_i(\tau) d\tau$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Roll angle</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Pitch angle</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Psi angle</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Euler-angles vector</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Position of the origin of the body-fixed frame, expressed in the inertial frame</td>
</tr>
<tr>
<td>$\Omega_i$</td>
<td>Angular velocity of the rotor around its axis</td>
</tr>
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</table>
\( f_i \)  
Force generated by the \( i \)th rotor

\( \mathbf{u} \)  
Vector \( \mathbf{u} \) of control inputs

\( \tau_{M_i} \)  
Drag force in the \( i \)th motor

\( \mathbf{f}_{i,d} \)  
Aerodynamic force vector

\( \tau_{\phi} \)  
Torque on roll motion

\( \tau_{\theta} \)  
Torque on pitch motion

\( \tau_{\psi} \)  
Torque on yaw motion

\( T \)  
Total thrust of the quadrotor

\( \mathbf{\tau}_a \)  
Applied torque vector to the quadrotor

\( \mathcal{B} \)  
Body-fixed frame

\( \mathcal{I} \)  
Inertial frame

\( ^p\mathbf{p} \)  
Position of the point \( \mathbf{p} \) with respect to the frame \( \mathcal{I} \)

\( \mathbf{v}_{\mathcal{B}} \)  
Linear velocity of the rigid body expressed in the body-fixed frame

\( \mathbf{v}_{\mathcal{I}} \)  
Linear velocity of the rigid body expressed in the inertial frame

\( ^\mathcal{B}\mathbf{R}_{\mathcal{I}} \)  
Orientation of frame \( \mathcal{B} \) with respect to \( \mathcal{I} \)

\( ^\mathcal{B}\mathbf{\omega}_{\mathcal{I}} \)  
Vector of absolute angular velocity of the rigid body with respect to \( \mathcal{I} \), expressed in the rigid body-fixed frame

\( \mathbf{\omega} \)  
Angular velocities vector of the quadrotor expressed in the body-fixed frame

\( \mathbf{W}_n \)  
Euler matrix expressed in the body-fixed frame

\( \mathbf{S}(\cdot) \)  
Skew-symmetric matrix of the vector \( \cdot \)

\( \text{vec} \cdot(\cdot) \)  
Operator that returns the vector from the skew-symmetric matrix \( \cdot \)

\( m \)  
Mass of the quadrotor UAV

\( l \)  
Distance between the rotors and the vehicle’s center of gravity

\( b \)  
Thrust coefficient of the rotors

\( k_r \)  
Drag coefficient of the rotors

\( g \)  
Gravitational acceleration
$I_{xx}$  Moment of inertia around the $x$-axis

$J$  Moment of inertia tensor

$b$  Unknown constant bias that represents the sum of the constant external disturbances, unmodeled dynamics and parametric uncertainties

$k_i$  Matrix of the gains of a controller
Introduction

In this introductory chapter the justification and objectives of this thesis are presented. It is organized in four sections: section 1.1 presents the Motivation; section 1.2 shows a State of the Art review; the Objectives are declared in section 1.3; and section 1.4 contains the subsequent chapters organization.

1.1 Motivation

Unmanned aerial vehicles\(^1\) (UAVs) have experienced extraordinary advancement in recent decades, being very useful in a wide variety of applications, especially when required to perform tasks in hazardous and inaccessible environments. Their use in industry has increased commercial potential with several civilian and military applications, including search and rescue, border surveillance, inspections, among others (Bouabdallah et al., 2004a; Bouabdallah and Siegwart, 2005; Valavanis, 2008; Serra et al., 2016).

As commented by Carrillo et al. (2012), during the 20th century, research and usage of UAVs were mostly for military proposes, with several applications. However, nowadays it is common civil roles for UAVs, as it can be viewed in new areas as ecological and environmental phenomena monitoring (Anderson and Gaston, 2013) and in remote sensing an mapping, with examples of use for low-altitude geospatial information and imaging.

---

\(^1\) Unmanned aircrafts are autonomous or remotely piloted aerial vehicles. They have been named in different ways and with varied acronyms, being the most used the Unmanned Aerial Vehicle (UAV), and others like Unmanned Aerial System (UAS), drone, aerial robots and Remotely-Piloted Aerial System (RPAS). In this thesis it will be adopted Unmanned Aerial Vehicle and the acronym UAV.
(Colomina and Molina, 2014). Moreover, new applications have been developed for UAVs, they have also been used with robotic manipulators (Mello et al., 2016; Abaunza et al., 2017), for aerial load transportation (Santos et al., 2017; Maza et al., 2009; Palunko et al., 2012; Rego and Raffo, 2016) and cooperative load transportation systems (Bernard et al., 2011).

The main applications can be summarized as:

- Images and video: Aerial video and photography for mapping, geological survey;
- Security: Disasters monitoring, incident control, missing persons search, remote monitoring;
- Agriculture: Plantation monitoring, peste control;
- Environmental and sustainability: Forest fire detection, monitoring and control of pollution, ecological monitoring;
- Industry: Industrial inspection, hazardous area monitoring, load transportation.

The fact that these aircrafts are unmanned allows lower costs, no risk for the crew and increased maneuverability on these aircrafts (Castillo et al., 2006). Among the different models of aircrafts, three types are the most used on unmanned flights, fixed-wings, rotary-wings and hybrid models.

Fixed-wing aircrafts are traditional aircrafts, on which the aerodynamic force is generated by the wings in the forward movement. This model presents lower energy consumption and enables long range missions. The group of rotary-wings aircrafts is composed of helicopters and their variants. In this kind of aircraft, the rotation of the wings generates lift by the air flow through the blades’ spin, which allows vertical movements and greater maneuverability. The hybrid models, e.g., tilt-rotors, mix features of rotary and fixed wings. Usually with two propellers, they have rotary wings with vertical initial position, like helicopters, but they tilt to forward position when in cruise flight. These three classes of aerial vehicles are presented in Figures 1.1, with a fixed-wing aircraft model FAB T-27 Tucano, 1.2, a quadrotor model, and 1.3, the tilt-rotor Bell-Boeing V-22 Osprey.

Figure 1.1: T-27 Tucano FAB (2013) Figure 1.2: AscTec Hummingbird Figure 1.3: Bell-Boeing V-22 Osprey Vertipedia (2018)
1.1. MOTIVATION

Helicopters are vertical take-off and landing (VTOL) aerial vehicles. This characteristic is an advantage over fixed-wing aircrafts, since it provides capabilities to take-off in small places, perform lateral and longitudinal flight, hover and fly in low altitudes (Valavanis, 2008). The quadrotor aircrafts constitute a particular type of helicopters with multi rotors, which possess advantages over the traditional ones when analyzed in terms of safety and efficiency at small scale. The main difference of quadrotors to traditional helicopters is due to the arrangement of the rotors. Quadrotors contain four rotors in X format (cross configuration), in which each of the propellers in the same axis turn in the same directions, opposite to the direction of the propellers on the other axis. The vehicle control is performed by varying the motors velocities. Thereby, they are simpler helicopters without the complicated mechanical construction of swashplates and mechanical linkages found in conventional ones. Besides, they are still robust, insofar as they have maneuverability (Pounds et al., 2010).

The first quadrotor registered, and also the first human transport helicopter, is from beginning of the 20th century. In 1907, the brothers Louis Bréguet and Jacques Bréguet, under supervision of Professor Charles Richet, constructed the Bréguet Richet Gyroplane No. 1. Designed in a cross shape, it had four rotors and eight propellers, as showed in figure 1.4, and performed its first flight demonstration in the same year (Raza and Gueaieb, 2010). Another precursor of the four rotors helicopters was the Bothezat helicopter, illustrated in figure 1.5, which was built in 1920s as an experimental helicopter for the United States Army, with six blades in each rotor, having flown successfully (Seddon and Newman, 2011).

![Figure 1.4: Bréguet Richet Gyroplane No. 1](image)

After the experiments for human transportation, quadrotor’s efforts have been concentrated in unmanned ones, and its development have grown together with other configuration of UAVs. The pioneers UAVs were designed for military purposes, with first uses along World War I. Some decades later, VTOL aircrafts arouse more interest in civilian uses (Valavanis, 2008). Development in microprocessors, sensors, battery technologies, electronics compounds in general and the great advances in computer science in the last decades have spurred size reduction of processors and on-board hardware for a wide range of UAVs (Raptis and Valavanis, 2010).

Due to the development on technologies and construction price fall, combined to the
CHAPTER 1. INTRODUCTION

Figure 1.5: Bothezat helicopter Seddon and Newman (2011)

underactuated and unstable dynamics that have been substantially studied by robotics and control research groups, quadrotors have grown into one of the most used configurations for aerial robotics (Kai et al., 2017). Including the fact that several research groups have begun constructing quadrotor UAVs as robotics research tools (Hamel et al., 2002; Huang et al., 2009). Several recent works on the control research community have shown increasing interest in quadrotors UAVs, as it can be seen in Bangura and Mahony (2017); Kai et al. (2017); Muñoz et al. (2017); Serra et al. (2016); Hua et al. (2015); Carrillo et al. (2014); Mahony et al. (2012); Mellinger et al. (2012); Mellinger and Kumar (2011); Pounds et al. (2010); Raffo et al. (2008); Bouabdallah and Siegwart (2007); Hoffmann et al. (2007); Tayebi and McGilvray (2006).

As rotorcrafts in general, quadrotors are nonlinear underactuated mechanical systems with unstable dynamics, features that make controller design a challenging problem (Valavanis, 2008). Nonlinear control theory and applications are the main focus of much recent research within the control community, and this is a requirement in many applications, since real physical system are intrinsically nonlinear (Slotine et al., 1991).

Backstepping control technique is a constructive nonlinear control method, which interlaces Lyapunov functions in the feedback control design through a recursive method. The control design separates the problem in lower order subsystems, for which control Lyapunov functions are chosen to ensure a stability condition. According to Khalil (2002), this feature of breaking the problem in lower order subsystems offers flexibility that can be explored to solve robust control, tracking and stabilization problems under less restrictive conditions if compared to other techniques. In comparison with feedback linearization methods, backstepping enables to avoid useful nonlinearities cancellations providing the choice retaining beneficial ones, besides making possible flexibility in the design of the final control law (Krstic et al., 1995).

UAV’s control strategies have reached wide advances. However, some challenges when considering parametric uncertainties in the control design formulation still remain opened (Olavo et al., 2018). As commented in Lee (2013), stable control strategies designed without consideration of parametric uncertainties or errors on dynamics modeling, may
become unstable in the presence of small disturbances. To deal with this issue, the use of integral action in the backstepping control design is a proper technique for path tracking in the presence of constant disturbance, parametric uncertainties and unmodeled dynamics, even for time varying trajectories (Raffo et al., 2015; Skjetne and Fossen, 2004).

1.2 State of the art

In this section a literature review of recent and important works related to the development and improvement of UAVs, with special attention to quadrotors, and their control strategies are presented.

UAVs have experienced great popularity and considerable growth for diverse applications, mainly in function of technological progress on the construction of the aircrafts, like material engineering improvement, on embedded electronics and sensors advances, and batteries size reduction. Following the technological advances, size reduction of the UAVs are also an actuality, prototypes of small size quadrotors with the objectives of greater agility and ability, and lower power consumption for industrial applications are being developed (Kushleyev et al., 2013; Doshi et al., 2015; Thomas et al., 2017). Besides, improvement of the UAVs sensing, with advancement in visual detection and localization is another recent research focus. For instance landing quadrotor in a moving target (Serra et al., 2016) and quadrotor collision avoidance with obstacle detection (Alvarez et al., 2016; Roelofsen et al., 2015; Carrillo et al., 2015).

In face of the hardware development, specially due to new applications of the quadrotors, researches on control strategies are similarly necessary and important. The objective in control theory is to design stable closed loop systems, and for nonlinear systems the challenge is considerably complex. Despite of the great amount of control strategies designed for quadrotor UAVs, it remains a subject of researches in order to improve flight performance. In the following subsections 1.2.1 and 1.2.2 a literature review of contributions in recent control strategies applied to quadrotors is presented, with special attention to backstepping control approach and stability analysis.

1.2.1 Quadrotors Control Strategies

In this subsection, a review on recent works with contributions on control strategies applied to quadrotors is presented.

A controller for autonomous flight based on a Lyapunov control function was developed in Bouabdallah et al. (2004a) to control the orientation of a quadrotor. In Bouabdallah et al. (2004b) two model-based control techniques for the quadrotor were compared. A classical Proportional-Integral-Derivative (PID), and a Linear Quadratic Regulator (LQR) applied with a linearized model. A PID controller is also used for stabilization of a
quadrotor in Bolandi et al. (2013). In this paper a linearized model of the quadrotor was used, on which the PID tuning was implemented through an optimization-based method, aiming disturbance rejection of the attitude subsystem.

Two control techniques were compared in Bouabdallah and Siegwart (2005), backstepping and sliding-mode. In the outer loop, the position control generates desired angles to the inner loop, where the controllers actuate on the control inputs. These controller were not able to reject sustained disturbances.

In Tayebi and McGilvray (2006), controllers were developed for quadrotor attitude stabilization. The control strategy was based on a proportional-double-derivative controller, which included Coriolis and gyroscopic torques compensation. The proportional and the two derivative actions were implemented in the quaternion algebra space. This design gave global exponential stabilization. Another proposed controller was a proportional-derivative without compensation of the Coriolis and gyroscopic torques that provides global asymptotic stability. In this controller, the proportional action is in terms of the quaternions and the derivative action in terms of quadrotor body-fixed frame angular time derivative. The developed controllers were implemented experimentally in a small-scale quadrotor UAV.

Cascade control strategies, using two control loops, with an inner control loop for the orientation dynamics and an outer loop for controlling the position dynamics is common for quadrotor path tracking problem. This approach can be found in Bouabdallah and Siegwart (2007), where the position controllers (planar position) generate desired angles to the attitude controllers. The altitude and yaw angle are controlled with individual control for each one, which have the same control technique, an integral backstepping.

In Escareño et al. (2013), a controller for trajectory tracking of a quadrotor with two dimensional wind disturbance is presented. A backstepping controller is proposed to stabilize the inner loop, of the translational movements, and the second control loop is designed through a sliding-mode technique.

In Das et al. (2009), a nonlinear adaptive state feedback controller with dynamic inversion is introduced. An outer control loop with a proportional derivative controller, and an inner control loop with a feedback by linearizing control are designed. The stability and tracking performance are demonstrate by Lyapunov stability theory. A similar technique was proposed by Cabecinhas et al. (2014) for a quadrotor in a path following problem. The control solution rejects constant force disturbances while asymptotically stabilizes the closed-loop system. Goodarzi et al. (2015) presented another nonlinear adaptive control, used for a quadrotor tracking control under influence of parameter uncertainties.

An hierarchical controller for UAVs using singular perturbation theory and the stability analysis were presented in Bertrand et al. (2011). The position and attitude dynamics were considered with a time-scale separation between them, and this controller design was
implemented in a quadroto

1.2.2 Backstepping Control of Unmanned Aerial Vehicles and Quadrotors

In this subsection it is presented recent contributions on backstepping control approach applied to UAVs and quadrotors.

In order to control UAVs, many works in the literature are found using backstepping technique. Backstepping control method is a technique for cascade systems, that achieves stability through the design of control Lyapunov functions. Mahony and Hamel (2004) proposed a backstepping control design for trajectory tracking of a helicopter considering bounds on initial error and trajectory parameters. In Madani and Benallegue (2006a), a backstepping control design is presented, in which the dynamic model of the quadrotor was split in three subsystems. A first subsystem considered the planar position, from which the backstepping output is the desired pitch and roll angles. The second subsystem is formed with the altitude and yaw angle, and the third one considered the output of the first and the second subsystem to compose a four dimensional vector to backstepping the control inputs. In Madani and Benallegue (2006b), a similar backstepping controller is presented for path tracking, where the main difference is due to the outputs of the controller, that are the forces generated in each propeller of the quadrotor.

To control quadrotor UAVs, backstepping control technique is one of the most used techniques. However, it is usually applied to perform path tracking and stabilization problems separately, which can lead to stability issues of the whole system. Huang et al. (2010) implemented a backstepping control for the translational movements and the yaw angle orientation of a quadrotor. They considered the UAV mass as an unknown parameter.

In Raffo et al. (2008) a robust control was developed for the quadrotor path tracking problem. The control design was performed in two loops, with a $H_\infty$ controller in charge of the stabilization and a backstepping control law used to perform the translational path tracking.

A backstepping controller for a helicopter is developed in Raptis et al. (2011) by applying the properties of the rotation matrix, which are used to compute a vector of the desired orientation for the translational movements. The magnitude of this vector gives the necessary total thrust, and its direction gives the roll and pitch reference angles. The yaw angle is controlled with an independent backstepping control law, in order to achieve the necessary yaw torque input.

In Tan et al. (2016), a backstepping control approach was designed using contraction theory, in which the altitude dynamics are separated from the planar ones. Null steady-state error is achieved in the quadrotor path tracking for planar movements without integral action in the control design.
Integral action can be added in the backstepping control design to ensure null steady-state error in presence of parametric uncertainties, external disturbance and unmodeled dynamics. Skjetne and Fossen (2004) showed that, for a generic plant, adding integral action in the first step of the backstepping control design guarantees convergence for constant reference trajectory, but could not be guaranteed when the reference trajectory is time-varying. However, by adding the integral action in the second step of the backstepping control design, the convergence is achieved for both constant and time-varying signals. Adding the integral action in the second step means that it is integrating the velocity dynamics, while in the first step is about the position dynamics.

In Bouabdallah and Siegwart (2007), the integral action is included in the first step of a backstepping control design for a quadrotor. The formulation aimed the convergence of the errors for constant orientation references. Mian and Daobo (2008) also included an integral action in the first step of the backstepping control design, in the problem of convergence of the roll angle error. Jasim and Gu (2015) controlled translational path tracking of a quadrotor using backstepping control with integral action, also in the first step control.

The Uniform Global Asymptotically Stability (UGAS) for systems with the integral action in the second step of a backstepping control approach was designed in Skjetne and Fossen (2004), which was based on the Matrosov’s Theorem version presented in Loria et al. (2002). This control approach was used in Raffo et al. (2015), where a robust nonlinear control strategy is proposed for quadrotor path tracking, in which a backstepping controller with integral action for translational movements, with guarantees of stability and convergence of the tracking error for time-varying reference signal in the presence of disturbances, while a nonlinear $H_{\infty}$ controller stabilizes the rotational movements.

Lyapunov theory is widely used for analysis of nonlinear time-varying systems. Matrosov’s Theorem, firstly presented in Matrosov (1962), provides an uniform attractiveness condition of the origin for uniform stable nonlinear time-varying systems, as a complementary approach to the Lyapunov stability theory. It is a complementary stability analysis method of Lyapunov approach, with sufficient conditions for asymptotically stability proof without the need of a strict Lyapunov function. Uniform asymptotically stability is concluded with the employment of a non-strict Lyapunov function in combination with an auxiliary function, whose time derivative is strictly nonzero where the time derivative of the Lyapunov function is zero (Loria et al., 2002; Malisoff and Mazenc, 2009).

Some works showed variations of Matrosov’s Theorem. In Astolfi and Praly (2011) a simplified version of Matrosov’s Theorem is presented, with limited convergence, and in Loria et al. (2002) and Loria et al. (2005) Matrosov’s Theorem is extended to the use of multiple auxiliary functions, with global asymptotically stability. The proposed nested Matrosov Theorem of Loria et al. (2005) was used by Wang et al. (2012) for stability proof in a formation control problem of autonomous underwater vehicles, where passivity control
techniques were used to solve velocity reference tracking.

There are some applications of Matrosov’s Theorem in robotic systems, as it can be viewed in Paden and Panja (1988), where the authors used it to prove global stability of a PD+ controller in a position tracking problem for a robot manipulator. In Nicosia and Tomei (1995), a dynamic output feedback controller is designed for flexible joint robots trajectory tracking, guarantying asymptotic convergence.

In Qi et al. (2018), Matrosov’s Theorem was used to analyze the stability of a cooperative system with multiple quadrotors coupled each one with a robotic arm, called quadrotor-manipulator system. The controller was designed for tasks execution considering collision avoidance. The position reference for each pair of quadrotor-manipulator is provided by a following avoidance function, and the convergence of the robots to the trajectories is proved by Lyapunov theory and Matrosov’s theorem.

In Federal University of Minas Gerais (UFMG), recent researches have been developed related to these fields, as Pimenta et al. (2013) with decentralized controller for a swarm of aerial robots, development of UAV Iscold et al. (2010), solving problems of coverage path planning with groups of UAV Avellar et al. (2015) and multiple UAV trajectory planning Neto et al. (2009).

1.3 Objectives

The main goal in this work is to solve the problem of trajectory tracking of UAVs, considering the presence of unmodeled dynamics, external disturbances and parameter uncertainties, besides of guaranteeing stability of the whole closed-loop dynamics of the quadrotor UAV.

1.3.1 Specific Objectives

Specific objectives of this work are:

- Design backstepping nonlinear control to perform trajectory tracking of a quadrotor UAV, assuming its whole-body six degrees of freedom (6DOF) dynamics;

- Design control strategy for time-varying trajectory tracking of a quadrotor UAV with the presence of parametric uncertainties, unmodeled dynamics and constant external disturbances;

- Use rotation matrix properties in the control design to avoid singularities in the coupling of position and orientation dynamics of the quadrotor.
1.4 Text Structure

The structure of the manuscript is developed as follows:

- **Chapter 2**: Review of techniques used in the control formulation of this dissertation;
- **Chapter 3**: Quadrotor modeling based on the Newton-Euler approach;
- **Chapter 4**: Design of control strategies and presentation of the simulation results;
- **Chapter 5**: Discussion of conclusions of this dissertation, and suggestions of future works.

1.5 Publication

The following article was published during the development of this thesis:

This chapter introduces the nonlinear control techniques that will be employed throughout this thesis. In the first section it is presented the backstepping technique. In section 2.2, Matrosov’s Theorem is presented. Section 2.3 presents the backstepping formulation with integral action for time-varying reference signals convergence.

2.1 Standard Backstepping

This section describes the Backstepping control technique, which is a nonlinear control approach to stabilize the origin of systems in strict-feedback form, designed systematically by constructing feedback control laws through Control Lyapunov Functions (CLF).

This methodology divides the system model into subsystems, and the state variables are assumed as virtual inputs to the feedback control laws in the subsystems. This procedure is executed selecting these virtual inputs in a number of the split steps, lower or equal than the system order. It is initialized in the higher subsystem, aiming to control and stabilize this first subsystem, succeeding to the second higher one, as a recursive tool, until the last subsystem, where the inputs of the system actuate, is reached, and thereby creating a complete feedback control law (Krstic et al., 1995; Khalil, 2002).

Among the advantages of this method, two of them can be specially highlighted. The first is the constructive feature of the control that simplifies the choice of the control law
Chapter 2. Background on Backstepping Control Techniques

In this thesis, it will be considered the input-affine nonlinear system given by

\[ \dot{x} = f(x) + g(x)u, \quad x \in \mathbb{R}^n, \quad f(0, 0) = 0. \] (2.3)

In this case, the time-derivative of the Lyapunov function will become:

\[ \frac{\partial V}{\partial x}[f(x) + g(x)u] \leq -W(x) < 0. \] (2.4)

2.1.2 Integrator backstepping

Backstepping approach is applicable to cascade structure systems in a lower-triangular form. Its scheme allows to design stabilizing control laws for a special class of nonlinear
dynamical systems, using the cascade decomposition property of the model. The integrator backstepping is a preliminary stage of the general backstepping approach, in which the control input is in a simple cascade integrator of the system model. This concept, based on the formulations given by Krstic et al. (1995) and Khalil (2002), can be introduced by the following system

\[
\begin{align*}
\dot{x} &= f(x) + g(x)\xi, \\
\dot{\xi} &= u,
\end{align*}
\]

where \(x \in \mathbb{R}, \xi \in \mathbb{R}, f(x) : D \rightarrow \mathbb{R} \) and \(g(x) : D \rightarrow \mathbb{R} \) are known nonlinear functions, and \(g(x)\) is strictly nonzero. The variable \(u \in \mathbb{R}\) is the control input.

The relation between the two subsystems in (2.5) is illustrated in the Figure 2.1.

![Figure 2.1: Block diagram of the cascade integrator (2.5)](image-url)

The objective is to obtain a behavior of the control input \(u\) in (2.5b), such that the state variable \(\xi\) stabilizes the subsystem (2.5a) in the origin \(x = 0\). In this way, the procedure begins considering \(\phi = \xi\), where \(\phi(0) = 0\), as a virtual control input and evaluating the existence of a candidate Lyapunov function \(V_1(x)\) that satisfies

\[
\frac{\partial V_1}{\partial x}[f(x) + g(x)\phi(x)] \leq -W(x) < 0.
\]

An equivalent system of (2.5) can be represented by adding and subtracting \(g(x)\phi(x)\) in equation (2.5a), as follows

\[
\begin{align*}
\dot{x} &= [f(x) + g(x)\phi(x)] + g(x)[\xi - \phi(x)], \\
\dot{\xi} &= u.
\end{align*}
\]

Using the change of variables \(z = \xi - \phi\) and \(v = u - \dot{\phi}\), the system will be asymptotically stable for \(z = 0\). After, consider a second candidate Lyapunov function to evaluate the
two subsystems together, as follows
\[ V_2(x, \xi) = V_1(x) + \frac{1}{2} z^2, \quad (2.8) \]
whose time derivative is given by
\[ \dot{V}_2 = \frac{\partial V_1}{\partial x} [f(x) + g(x)\phi(x)] + \frac{\partial V_1}{\partial x} g(x)z + zv \leq -W(x) + \frac{\partial V_1}{\partial x} g(x)z + zv. \quad (2.9) \]

By choosing an appropriate \( v = -\frac{\partial V_1}{\partial x} g(x) - kz \), with \( k > 0 \) yields to \( \dot{V}_2 \leq -W(x) - k z^2 \). The complete state feedback control law is given by
\[ u = \frac{\partial \phi}{\partial x} [f(x) + g(x)\xi] - \frac{\partial V_1}{\partial x} g(x) - k[\xi - \phi(x)], \quad (2.10) \]
which leads the origin \([x = 0, \xi = 0]^T\) to be asymptotically stable.

### 2.1.3 Recursive Backstepping Design Procedure

The previous technique can be extended to deal with strict-feedback form systems, associating linearizing state feedback control laws (Feedback Linearization), as presented in Krstic et al. (1995); Khalil (2002). The following system represents a lower triangular one with dimension two,

\[ \begin{align*}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u, 
\end{align*} \quad (2.11) \]

where \( \dot{x}_i \) defines the time derivative of the state variables \( x_i \), with \( g_i \neq 0, \forall x_i \), \( u \) is the control input variable, and \( f_i \) and \( g_i \) are functions of class \( C^{n-1} \), where \( n \) is an integer number bigger than the number of steps of the backstepping procedure for the system (2.11).

This system can be viewed as a linkage of two separated subsystems where the first subsystem (2.11a) is represented in the diagram block of Figure 2.2, and the second subsystem (2.11a), that completes the cascade structure, is illustrated in the diagram block of Figure 2.3.

The control design for this system is obtained similarly to the *integrator backstepping* approach. Thus, if it is applied a feedback linearization control law, \( u \), in (2.11b) in the

---

1 A function \( h(x) \) is said to be of class \( C^1 \) if its first derivative \( h'(x) \) exist and is continuous. The function \( h(x) \) is said to be of class \( C^2 \) if the first and the second derivative \( h'(x), h''(x) \) exist and are continuous. The function \( h(x) \) is said to be of class \( C^n \) if the \( n \) first derivatives \( h'(x), h''(x), ..., h^n(x) \) exist and are continuous.
2.1. STANDARD BACKSTEPPING

The system (2.11) is rewritten exactly as the integrator backstepping (2.5):

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + g_1(x_1) x_2, \quad (2.13a) \\
\dot{x}_2 &= v_1. \quad (2.13b)
\end{align*}
\]

Following form

\[
u = \frac{1}{g_2(x_1, x_2)} [v_1 - f_2(x_1, x_2)], \quad (2.12)
\]
therefore, given a \textit{n-order} system in strict-feedback form

\begin{align}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3, \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4, \\
\vdots \\
\dot{x}_n &= f_n(x_1, x_2, ..., x_n) + g_n(x_1, x_2, ..., x_n)u, \\
\end{align}

(2.14)

where \( x \in \mathbb{R}^n \), the origin \([x_1, x_2, x_3, ..., x_n]^T = [0, 0, 0, ..., 0]^T \) is an equilibrium point, the functions \( f_1, f_2, f_3, ..., f_n \) disappear at the origin, \( x_2, x_3, ..., x_n \) are scalars, and all functions \( g_1, g_2, g_3, ..., g_n \neq 0 \).

The state \( x_3 \) can be selected as a virtual control input to stabilize the dynamic of \( x_2 \) in the second equation of (2.14), in the same backstepping procedure of (2.12)

\[ x_3 = \frac{1}{g_2(x_1, x_2)}[\phi_1 - f_2(x_1, x_2)], \]

(2.15)

and recursively for \( x_4 \) and the next states as virtual control inputs, until the input \( u \) in the last equation of (2.14), achieving closed-loop asymptotically stability with a complete CLF in the form:

\[ \dot{V}_n \leq -W(x_1) - k_1 x_2^2 - k_2 x_3^2 - ... - k_{n-1} x_n^2 < 0, \quad k_1, k_2, ..., k_{n-1} > 0. \]

(2.16)

The strict-feedback closed-loop form of the \textit{n-order} system (2.14) is illustrated in Figure 2.4.

### 2.2 Matrosov’s Theorem

In this section, a new version of the Matrosov’s Theorem, proposed by Loría et al. (2002) is presented. Initially, Matrosov’s Theorem was proposed in Matrosov (1962), which extends, in a certain manner, the Krasovskii-LaSalle’s invariance principle to the case of general nonautonomous nonlinear systems. The original Matrosov’s Theorem is a tool for establishing Uniform Attractivity in Uniform Stable nonautonomous nonlinear systems through the employment of one auxiliary function with a Lyapunov function that establishes Uniform Global Stability. Its importance resides in the property that systems in which origin is simultaneously Uniform Global Stable (UGS) and Uniform Global Attractivity will be UGAS (Loría et al., 2002).

Systems with UGS have a non-strict Lyapunov function associated to the trajectories evolution on time. The non-strict condition of the Lyapunov functions resides in the definition of a positive definite function with negative semi-definite derivative along all the
2.2. MATROSOV’S THEOREM

Figure 2.4: Closed loop of a \( n \)-order strict-feedback system
trajectories, i.e., \( W(x) > 0 \) and \( \dot{W}(x) \leq 0 \) for all \( x \in \mathbb{R} - \{0\} \). This is a weaker condition than a strict Lyapunov function, where its time-derivative along all the trajectories is negative definite, i.e., \( \dot{W}(x) < 0 \) for all \( x \in \mathbb{R} - \{0\} \).

Loría et al. (2002) presented an extension of the original Matrosov’s Theorem, on which the number of auxiliary functions, which are combined with a non-strict Lyapunov function, are increased from one to an arbitrary finite number.

Consider a nonautonomous nonlinear system given by

\[
\dot{x} = f(x, t),
\]

(2.17)

where the function \( f(x, t) \) is continuous in time and locally Lipschitz for the states, and assume that the system (2.17) is UGS. Matrosov’s Theorem establishes sufficient conditions to provide that the solution is also Asymptotically Stable. These conditions are stated as follows:

**Assumption 1.** The system (2.17) is UGS in the origin.

**Assumption 2.** There exist integers \( j, m > 0 \), and for all \( \Delta > 0 \) there exist: i) an scalar \( \mu > 0 \); ii) continuous and locally Lipschitz functions \( W_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, i \in \{1, \ldots, j\} \); iii) continuous functions \( Y_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, i \in \{1, \ldots, j\} \); and iv) a continuous function \( \varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m, i \in \{1, \ldots, m\} \), where for almost all \( (t, x) \in \mathbb{R} \times \beta(\Delta) \):

\[
\max\{|W_i(t, x)|, |\varphi(t, x)|\} \leq \mu,
\]

(2.18)

\[
\dot{W}_i(t, x) \leq Y_i(x, \varphi(t, x)).
\]

(2.19)

**Assumption 3.** For all integers \( k \in \{1, \ldots, j\} \), the condition:

a) \((z, \psi) \in \beta(\Delta) \times \beta(\mu), Y_i(z, \psi) = 0 \) for all \( i \in 1, \ldots, k - 1 \);

implies

b) \( \{Y_i(z, \psi) \leq 0\} \).

**Assumption 4.** The condition:

a) \((z, \psi) \in \beta(\Delta) \times \beta(\mu), Y_i(z, \psi) = 0 \) for all \( i \in 1, \ldots, j \);

implies

b) \( \{z = 0\} \).

**Theorem 1.** (Loría et al. (2002)): Under the assumptions 1, 2, 3 and 4, the system (2.17) is Uniform Global Asymptotically Stable (UGAS) in the origin.

The assumption 1 gives the UGS condition and the assumptions 2, 3 and 4 establish Uniform Attractivity of the origin, sufficient conditions to generalize the Matrosov’s Theorem for UGAS of the origin for nonautonomous nonlinear systems with \( m \) auxiliary functions. The proof of stability of Theorem 1 is given in Loría et al. (2002).
2.3 Integral Action for Path Tracking

In control theory, integral action has the properties of tracking constant or slowly varying reference signals with null steady-state errors, rejecting constant external disturbances (Skogestad and Postlethwaite, 2005; Åström and Hägglund, 1995). In practice, it is a robustifying part of the feedback controller, also mitigating problems with parametric uncertainties and unmodeled dynamics (Skjetne and Fossen, 2004).

When dealing with the backstepping control design, integral action can be added in the first step of error dynamics in order to attenuate the effect of constant reference trajectories (Bouabdallah and Siegwart, 2007). Adding the integral action in the second step means that it is integrating the velocity dynamics errors, while the first step is about the position dynamics. The use of the integral action in the second step of the backstepping approach guarantees convergence for both constant and time-varying reference signals, as stated in Skjetne and Fossen (2004) and applied by Raffo et al. (2015).

In this section, it is presented the backstepping with integral action for path tracking and also is analyzed the stability of the controller in presence of constant external disturbances.

Consider the following nonlinear system of relative degree 2 given by

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2, \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u + d,
\end{align*}
\]

(2.20)

where \(x_1, x_2 \in \mathbb{R}^n\) are the states, the origin \([x_1, x_2]^T = [0, 0]^T\) is an equilibrium point, \(f_1, f_2, g_1, g_2\) are smooth functions with \(g_1, g_2 \neq 0\), and \(u\) is the control input vector. The variable \(d\) is a constant bias that corresponds to the sum of all parametric uncertainties, external disturbances and unmodeled dynamics.

Aiming to solve the tracking problem for the system (2.20), the control objective is to track a reference time-varying trajectory vector \(x_r(t)\), from which the error vector can be defined as

\[
\mathcal{E}_1 = x_1(t) - x_r(t),
\]

(2.21)

and the control problem now is to ensure that \(\lim_{t \to \infty} \mathcal{E}_1(t) = 0\). The error dynamics of (2.21) are \(\dot{\mathcal{E}}_1 = \dot{x}_1(t) - \dot{x}_r(t)\), from which the system (2.20) can be rewritten as:

\[
\begin{align*}
\dot{\mathcal{E}}_1 &= f_1(x_1) + g_1(x_1)x_2 - \dot{x}_r(t) \quad (2.22a) \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u + d. \quad (2.22b)
\end{align*}
\]

Therefore, the backstepping technique can be applied to control the system (2.22) through the following steps:

- **Step 1**
CHAPTER 2. BACKGROUND ON BACKSTEPPING CONTROL TECHNIQUES

The procedure of designing the feedback control is started with a virtual control input $x_2 = \phi_1(E)$, and the first Lyapunov function candidate $V_1$ chosen as

$$V_1(E_1) = \frac{1}{2} E_1^T E_1. \quad (2.23)$$

Its time derivative is given by

$$\dot{V}_1(E_1) = E_1^T \dot{E}_1 \quad (2.24)$$

from where the virtual control input is designed as follows

$$\phi_1(E_1) = \frac{1}{g_1(x_1)} [-f_1(x_1) + \dot{x}_r(t) - k_1 E_1], \quad (2.25)$$

where $k_1$ is a diagonal square matrix of positive values as controller gain. Applying (2.25) in (2.24), it is obtained the time derivative of the Lyapunov function

$$\dot{V}_1(E_1) = -E_1^T k_1 E < 0, \quad (2.26)$$

which is definite-negative.

By adding and subtracting $g(x)\phi_1$ in (2.22a), the following dynamics are obtained

$$\dot{E}_1 = -k_1 E_1 + g_1(x_1) [x_2 - \phi_1], \quad (2.27)$$

which ensures that $E_1$ will converge asymptotically to the origin if $x_2 - \phi_1 = 0$.

- Step 2

In the second step of the backstepping control, the behavior of $x_2 - \phi_1$ is analyzed assuming the change of variables $E_2 = x_2 - \phi_1$, from which the control problem pursues the objective $\lim_{t \to \infty} E_2(t) = 0$. As explained in Skjetne and Fossen (2004), a generic plant with the addition of integral action in the first step of the backstepping control can not guarantee convergence for time-varying reference signals. However, if the integral term is included in the second step of this control approach, convergence for both constant and time-varying reference signals are guaranteed. Adhering to this concept, the control problem, in this stage, will be redesigned with the dynamics of the new variable $E_2$ with the integral term $\dot{X} = \int_0^t E_2(\tau)d\tau$. Thus, by rewriting (2.22a) yields

$$\dot{E}_1 = -k_1 E_1 + g_1(x_1) E_2, \quad (2.28a)$$

$$\dot{X} = E_2, \quad (2.28b)$$

$$\dot{E}_2 = f_2(x_1, x_2) + g_2(x_1, x_2) u + d - \dot{\phi}_1. \quad (2.28c)$$
2.3. INTEGRAL ACTION FOR PATH TRACKING

This system keeps a strict-feedback form and have relative degree increased from 2 to 3. Moreover, the Lyapunov function candidate

\[ V_2(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) = V_1 + \frac{1}{2} \mathcal{X}^T k_\mathcal{X} \mathcal{X} + \frac{1}{2} \mathcal{E}_2^T \mathcal{E}_2, \tag{2.29} \]

with \( k_\mathcal{X} = k_\mathcal{X}^T > 0 \), is enough to analyze the stability of the dynamic equations (2.28). Its time-derivative is given by

\[ \dot{V}_2(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) = -\mathcal{E}_1^T k_1 \mathcal{E}_1 + \mathcal{E}_1^T g_1(x_1) \mathcal{E}_2 + \mathcal{X}^T k_\mathcal{X} \mathcal{X} + \mathcal{E}_2^T [f_2(x_1, x_2) + g_2(x_1, x_2)u + d - \dot{\varphi}_1], \tag{2.30} \]

and with the choice of the feedback control law \( u \) as follows

\[ u = \frac{1}{g_2(x_1, x_2)} [-f_2(x_1, x_2) + \dot{\varphi}_1 - k_2 \mathcal{E}_2 - g_1^T(x_1) \mathcal{E}_1 - k_\mathcal{X} \mathcal{X}], \tag{2.31} \]

where \( k_2 > 0 \), makes (2.30) becomes

\[ \dot{V}_2 = -\mathcal{E}_1^T k_1 \mathcal{E}_1 - \mathcal{E}_2^T k_2 \mathcal{E}_2 + \mathcal{E}_2^T d, \tag{2.32} \]

which is negative definite for \( d = 0 \). Rewritten (2.28) with the control input (2.31), the new system is given by

\[ \dot{\mathcal{E}}_1 = -k_1 \mathcal{E}_1 + g_1(x_1) \mathcal{E}_2 \]

\[ \dot{\mathcal{X}} = \mathcal{E}_2 \]

\[ \dot{\mathcal{E}}_2 = -k_2 \mathcal{E}_2 - k_\mathcal{X} \mathcal{X} - g_1^T(x_1) \mathcal{E}_1 + d. \tag{2.33} \]

In the case of \( d = 0 \), the closed-loop system (2.33) has an equilibrium point in \([\mathcal{X}, \mathcal{E}_1, \mathcal{E}_2]^T = [0, 0, 0]^T\), which is Uniform Global Stable (UGS), but is not guaranteed the Uniform Global Asymptotically Stable (UGAS). If \( d \neq 0 \), \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) will keep the equilibrium point in 0, but the equilibrium point of \( \mathcal{X} \) will be shifted to \(-k_\mathcal{X} \mathcal{X} + d = 0\), what renders \([\mathcal{X}, \mathcal{E}_1, \mathcal{E}_2]^T = [k_\mathcal{X}^{-1}d, 0, 0]^T\).

Analyzing the stability of (2.33) by Matrosov’s Theorem, the origin \([\mathcal{X}, \mathcal{E}_1, \mathcal{E}_2]^T = [0, 0, 0]^T\) is UGS for the case of \( d = 0 \), meeting Assumption 1. Selecting appropriate continuous and bounded function \( \varphi(x_1(t), x_{1i}(t)) \) of the system states, for bounded and continuous \( x_1(t) \) and \( x_{1i}(t) \), and the following functions

\[ W_1(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) := V_2, \tag{2.34a} \]

\[ W_2(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) := \mathcal{X}^T \mathcal{E}_2, \tag{2.34b} \]

\[ Y_1(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) := -\mathcal{E}_1^T k_1 \mathcal{E}_1 - \mathcal{E}_2^T k_2 \mathcal{E}_2, \tag{2.34c} \]

\[ Y_2(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) := \mathcal{E}_2^T \mathcal{E}_2 - \mathcal{X}^T g_1(x_1)^T \mathcal{E}_1 - \mathcal{X}^T k_\mathcal{X} \mathcal{X} - \mathcal{X}^T k_\mathcal{X} \mathcal{E}_2, \tag{2.34d} \]
for bounded \((\mathcal{E}_1, \mathcal{E}_2, \mathcal{X})\), the functions \(W_1(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X})\) and \(W_2(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X})\) are bounded, what meets to the Assumption 2. Moreover, if \(Y_1(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) = 0\), implies \(Y_2(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) < 0\), according to Assumption 3. Furthermore, in case of \(Y_1(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) = Y_2(\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) = 0\), implies the states \((\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) = 0\), in conformity to Assumption 4. What proves that the origin \((\mathcal{E}_1, \mathcal{E}_2, \mathcal{X}) = 0\) is UGAS.

Evaluating the case that the bias \(d \neq 0\), the system (2.33) can be rewritten defining \(\tilde{\mathcal{X}} = \mathcal{X} - k_x^{-1}d\)

\[
\begin{align*}
\dot{\mathcal{E}}_1 &= -k_1 \mathcal{E}_1 + g_1(x_1) \mathcal{E}_2 \\
\dot{\mathcal{X}} &= \mathcal{E}_2 \\
\dot{\mathcal{E}}_2 &= -k_2 \mathcal{E}_2 - k_x \tilde{\mathcal{X}} - \frac{g_1^T(x_1) \mathcal{E}_1}{2}
\end{align*}
\] (2.35)

which is the same as (2.33) for \(d = 0\), proving that (2.33) is UGAS for any constant bias \(d\).

### 2.4 Chapter Conclusions

In this chapter, the nonlinear control techniques used in the controllers design in this thesis have been described. It is introduced the Backstepping control approach, including the CLF concept, the Integrator Backstepping and the Recursive Backstepping for high order systems. It is described a recent version of the Matrosov’s Theorem, which is appropriate for asymptotically stability analysis of nonlinear systems. Furthermore, it is introduced the Integral Action formulation for disturbance rejection in time-varying trajectory tracking.
In this chapter it is described the quadrotor UAV, with the development of its equations of motion. The helicopter dynamic model is obtained under Newton-Euler mathematical formulations, based on Raffo (2011) and references therein.

3.1 Introduction

Quadrotor UAV is an aerial vehicle with six degrees of freedom (DOF), from which three of them are the positions in the Euclidean space and the three orientation angles. This model has four independent propellers that generate the four control inputs, which enable to perform tracking of four of the six DOF, characterizing this vehicle as an underactuated mechanical system.

An important construction feature of this aircraft is the configuration with the four coplanar propellers in a cross shape. Each propeller is a rotating wing formed by an assemblage of an electrical motor with a rotor, which produces a force \( f_i, \ i = 1...4 \), given by:

\[
f_i = b\Omega_i^2,
\]

where \( \Omega_i \) is the angular velocity of the rotor around its axis, and \( b \) is the thrust coefficient of the rotors. These actuators operate in two pairs spinning in opposite directions, the motors 1 and 3 rotate in the same direction, clockwise for example, while 2 and 4 in the
opposite direction, in this case counterclockwise.

The force generated $f_i$ in the $i$ rotor’s center produces a momentum on the vehicle’s center of gravity, which quantity is given by the multiplication of this force $f_i$ by the distance $l$ between these two points. The contribution of the momentum caused by the force $f_2$ minus the one produced by the force $f_1$ provokes the torque $\tau_\phi$ on roll motion. The torque on pitch motion is achieved similarly through the difference of the momentums generated by the forces $f_3$ and $f_1$.

Furthermore, each electrical motor generate a torsion effort $\tau_{M_i}$ while spinning, the yaw motion forced by torque generated from the sum of this torsion effort in the four motors (Raffo, 2011). The three torques generated by the control inputs are given by

$$\begin{bmatrix}
\tau_\phi \\
\tau_\theta \\
\tau_\psi
\end{bmatrix} = \begin{bmatrix}
l(f_2 - f_4) \\
l(f_3 - f_1) \\
\sum_{i=1}^{4} \tau_{M_i}
\end{bmatrix}. \quad (3.2)$$

Figure 3.1 illustrate the actuation of the quadrotor’s torques.

![Figure 3.1: Forces and torques of the Quadrotor](image)

Assuming the position of the motor 1 as the front side of the quadrotor on Figure 3.1, and the motor 2 as the left side, the roll torque, $\tau_\phi$, generates lateral motions, tilting the quadrotor body in left and right movements. In the same manner, the pitch motion tilts the quadrotor forward and backward, and therefore longitudinal displacements, by way of the torque $\tau_\theta$.

The yaw movement is defined as the rotation around the perpendicular axis that cross the plane formed by the four propellers. When the four motors velocity are the same, the counter-torque produced by the pair $(f_1, f_3)$ is canceled by the other pair $(f_2, f_4)$ and the
yaw torque is null. This is an important difference between this type of aerial vehicle and traditional helicopters.

The aerodynamic drag force in each motor, given by $\tau_{\text{drag}} = k_r \Omega_i^2$, with a constant $k_r > 0$, is an opposite force to the torsion effort. Assuming each motor in steady-state and taking $\tau_{M_i} = \tau_{\text{drag}}$ and equation (3.1), the applied torques (3.2) can be rewritten according to Raffo (2011) as

$$
\begin{bmatrix}
\tau_{\phi_a} \\
\tau_{\theta_a} \\
\tau_{\psi_a}
\end{bmatrix} =
\begin{bmatrix}
lb(\Omega_2^2 - \Omega_4^2) \\
lb(\Omega_3^2 - \Omega_1^2) \\
k_r(\Omega_1^2 - \Omega_2^2 + \Omega_3^2 - \Omega_4^2)
\end{bmatrix}.
$$

(3.3)

The sum of the four propellers’ forces moves the quadrotors in the axis perpendicular to the plane of the propellers, this force is called the total thrust $T$ and is given by (Castillo et al., 2006)

$$
T = \sum_{i=1}^{4} f_i = \sum_{i=1}^{4} lb\Omega_i^2.
$$

(3.4)

The quadrotor is a lightweight flight system, thus both four propellers rotation and rigid body rotation in space cause gyroscopic effects. The gyroscopic effects will be neglected in the dynamic model of this thesis, and their influence will be assumed as disturbances in the whole body rotation, as in Raffo (2011).

### 3.2 Rigid Transformations

In this section it is presented the kinematics for rigid-body mechanical systems, including firstly the introduction of vehicle’s rigid transformations in the three-dimensional Euclidean space.

The quadrotor is assumed as a rigid-body vehicle with a frame linked to it, represented by $\mathcal{B} = \{B_1, B_2, B_3\}$, where $B_1$ is the front quadrotor flight direction, $B_2$ is in the horizontal plane and orthogonal to $B_1$ oriented with positive increasing to the left side (port), and $B_3$ is the axis orthogonal to the plane $B_1 o B_2$ with upward orientation. Furthermore $\mathcal{I}$ is the inertial frame considered fixed to the Earth. The position of the mass center of the quadrotor in the inertial frame is given by $\xi = [x \ y \ z]^T \in \mathbb{R}^3$. The Euler angles, $\eta = [\phi \ \theta \ \psi]^T \in SE(3)$, represent the quadrotor orientation in the Euclidean space respect to the fixed frame $\mathcal{B}$. Figure 3.2 illustrates the quadrotor’s coordinate frames.

According to Spong et al. (2006), a rigid motion between different frames can be described by means of operations of rotation and translation. The coordinates of a point
\( \mathbf{p} \) rigidly attached to the frame \( \mathcal{B} \) can be expressed in frame \( \mathcal{I} \) as:

\[
\mathcal{I} \mathbf{p} = \mathcal{I} \mathbf{R}_{\mathcal{B}} \mathcal{B} \mathbf{p} + \mathcal{I} \xi,
\]

where \( \mathcal{I} \mathbf{p} \) and \( \mathcal{B} \mathbf{p} \) specify the position of the point \( \mathbf{p} \) with respect to the frames \( \mathcal{I} \) and \( \mathcal{B} \) respectively, \( \mathcal{I} \xi \) specifies the position of the origin of the quadrotor rotation frame expressed in the inertial frame \( \mathcal{I} \), and the rotation matrix \( \mathcal{I} \mathbf{R}_{\mathcal{B}} \) describes the orientation of frame \( \mathcal{B} \) with respect to \( \mathcal{I} \).

In Euler angles representation, the rotation matrix is obtained through three successive rotations around the axes of the fixed body frame, where a common convention is the rotation around \( x - y - z \), named roll, pitch and yaw representation, which is usually used for aerial vehicle applications, and also known as Tait-Bryan convention (Raffo, 2011). This rigid body rotation configuration in the space can be formulated as in Murray (1994):

i) Considering the initial position where the inertial and the body-fixed frames coincide, the first rotation is around \( B_3 \) axis by the yaw angle, \( \psi \):

\[
\mathbf{R}(B_3, \psi) = \begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

(3.6)

ii) The second movement is performed by a pitch motion around the new \( B_2 \) axis:

\[
\mathbf{R}(B_2, \theta) = \begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix}.
\]

(3.7)
iii) And the third movement is a rotation around the new \( B_1 \) by the \( \textit{roll} \) angle \( \phi \), that moves the rotorcraft to the final position:

\[
R(B_1, \phi) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix}.
\tag{3.8}
\]

The Euler angles are bounded: roll angle by \((-\pi < \phi < \pi)\), pitch angle by \((-\pi/2 < \theta < \pi/2)\), and yaw angle by \((-\pi < \psi < \pi)\).

Applying the three sequential rotations above, the rotation matrix from the inertial frame \( I \) to the fixed body frame \( B \) expressed by

\[
\mathcal{R}_I^B = R(B_3, \psi) \cdot R(B_2, \theta) \cdot R(B_1, \phi)
\]

\[
\mathcal{R}_I^B = \begin{bmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix},
\tag{3.9}
\]

The rotational transformation that represents the orientation of the rigid body frame \( \mathcal{B} \) with respect to the inertial frame \( \mathcal{I} \) is given by \( (\mathcal{R}_I^B)^T = \mathcal{R}_B^I \), yielding to:

\[
\mathcal{R}_B^I = \begin{bmatrix}
\cos(\psi) \cos(\theta) & \sin(\psi) \cos(\theta) & -\sin(\theta) \\
\cos(\psi) \sin(\theta) \sin(\phi) - \sin(\psi) \cos(\phi) & \sin(\psi) \sin(\theta) \cos(\phi) + \cos(\psi) \cos(\phi) & \cos(\theta) \sin(\phi) \\
\cos(\psi) \sin(\theta) \cos(\phi) + \sin(\psi) \sin(\phi) & \sin(\psi) \sin(\theta) \cos(\phi) - \cos(\psi) \sin(\phi) & \cos(\theta) \cos(\phi)
\end{bmatrix}.
\tag{3.10}
\]

3.3 Quadrotor Kinematics

The kinematics equations of the quadrotor can be obtained by analyzing the time evolution of the translational movements (3.5) and rotational movements (3.10). The rotational kinematics can be evaluated through the time derivative of the rotation matrix (3.10). Considering that a rotation matrix \( R \in \mathbb{R}^{3 \times 3} \) is orthonormal, it implies the following properties:

- \( R^{-1} = R^T \),
- \( \det R = 1 \),
- \( R^T R = I_{3 \times 3} \),
where $I_{3 \times 3} \in \mathbb{R}^{3 \times 3}$ is the identity matrix. The time derivative of the third relation is given as

$$\dot{R}^T R + R^T \dot{R} = O_{3 \times 3}, \quad (3.11)$$

where $O_{3 \times 3} \in \mathbb{R}^{3 \times 3}$ is the zeros matrix. Let $S(\omega) \in SO(3)$ such that $S(\omega) = R^T \dot{R}$, then, equation (3.11) can be rewritten as:

$$S(\omega)^T + S(\omega) = O_{3 \times 3}. \quad (3.12)$$

In this case, $S(\omega)$ is a skew-symmetric matrix, which obeys the operation $(S(\omega)(\cdot) = \omega \times \cdot)$, where $\times$ denotes the cross product operator between two vectors and $(\cdot)$ represents a generic vector. The skew-symmetric matrix $S(\omega)$ for a vector $\omega = [\omega_1 \omega_2 \omega_3]^T$ is given by:

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad (3.13)$$

By multiplying $R^T \dot{R} = S(\omega)$ by $R$ from the left side and using the property $R^T R = I_{3 \times 3}$, it is obtained the equation:

$$\dot{R} = RS(\omega). \quad (3.14)$$

By analyzing the equation (3.14), the kinematic equation of the orientation of the rigid body expressed in the inertial frame $\mathcal{F}$ will be given as:

$$\dot{^\mathcal{F}R_{\mathcal{B}}} = ^\mathcal{F}R_{\mathcal{B}} S(^\mathcal{F}\omega_{\mathcal{B}}), \quad (3.15)$$

where $^\mathcal{F}\omega_{\mathcal{B}} = [p \quad q \quad r]^T$ is the vector of the absolute angular velocity of the rigid body with respect to $\mathcal{F}$, expressed in the rigid body-fixed frame.

The angular velocities of the body-fixed frame can be related to the time derivative of the Euler angles. The $z$ axis of the body-fixed frame is assumed the same of the inertial frame, therefore it is applied a first roll rates, followed by pitch rates with a rotation around $x$ axis velocities and yaw rates with a double velocity rotation around $y$ and $x$ axis, given in Raffo (2011) as follows:

$$\begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} = I_{3 \times 3} \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix} + R(x, \phi)^{-1} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} + (R(y, \theta)R(x, \phi))^{-1} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}, \quad (3.16)$$
resulting in:

\[
\begin{bmatrix}
0 & -\sin \theta & 0 \\
\cos \phi & \sin \phi \cos \theta & 0 \\
-\sin \phi & \cos \phi \cos \theta & 0
\end{bmatrix}
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & -\sin \theta \\
0 & \cos \phi & \sin \phi \cos \theta \\
0 & -\sin \phi & \cos \phi \cos \theta
\end{bmatrix}
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix}.
\] (3.17)

Denoting \( \eta = [\phi \ \theta \ \psi]^T \), this equation can be represented by:

\[
\omega = W_\eta \dot{\eta},
\] (3.18)

where \( W_\eta \) is the Euler matrix.

The angular rates of the body-fixed frame \( ^B\omega_{\mathcal{B}} \) can be measured in real aerial vehicles. The Euler angles time-derivative \( [\dot{\phi} \ \dot{\theta} \ \dot{\psi}]^T \) are obtained through the following relation

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix},
\] (3.19)

where the time derivative of the Euler angles is a discontinuous function due to the singularities of these angles.

The evolution of the translational movements can be expressed by the time derivative of (3.5), which means the linear velocity in \( \mathbb{R}^3 \) of a point \( p \) rigid attached to the frame \( \mathcal{B} \), expressed in the inertial frame \( \mathcal{I} \), as follows:

\[
^p\dot{p} = ^p\dot{R}_{\mathcal{B}}^\mathcal{B} p + ^pR_{\mathcal{B}}^\mathcal{B} \dot{p} + ^p\dot{\xi}.
\] (3.20)

Once the point \( p \) is rigid attached to the fixed-body frame \( \mathcal{B} \), its time derivative with respect to the its own frame is zero \( (^p\dot{p} = 0) \). Using (3.14) and adopting the notation \( v_\mathcal{B} = ^p\dot{p} \), the translational velocity is given by

\[
v_\mathcal{B} = ^pR_{\mathcal{B}}S(\omega)^{\mathcal{B}} p + ^p\dot{\xi}.
\] (3.21)

The linear velocity of the point \( p \) expressed in the fixed-body frame \( \mathcal{B} \) is obtained as follows

\[
v_{\mathcal{B}} = ^pR_{\mathcal{B}}v_\mathcal{B} = ^Bv_\mathcal{B},
\] (3.22)

where \( v_\mathcal{B} = [u_\mathcal{B} \ v_\mathcal{B} \ w_\mathcal{B}]^T \) and \( v_{\mathcal{B}} = [u_{\mathcal{B}} \ v_{\mathcal{B}} \ w_{\mathcal{B}}]^T \).
3.4 Newton-Euler Formulation of the Quadrotor

In this section it is presented the quadrotor equations of motion via Newton-Euler approach, considering the state vector \( \xi \) = \( [v, \eta, \omega]^T \), according to Raffo (2011).

The dynamic equations of a rigid body subject to external forces applied to the center of mass and expressed in the body-fixed frame can be obtained through the Newton-Euler approach using the relations (3.14) and (3.22) as given by

\[
\begin{align*}
    \begin{bmatrix}
        mI_{x3} & O_{x3} & J \\
        O_{x3} & J \\
        O_{x3} & O_{x3} & J
    \end{bmatrix}
    \begin{bmatrix}
        \dot{\xi} \\
        \dot{\omega}
    \end{bmatrix}
    +
    \begin{bmatrix}
        \beta \omega \times m v \\
        \beta \omega \times J \omega
    \end{bmatrix}
    =
    \begin{bmatrix}
        f \\
        \tau
    \end{bmatrix},
\end{align*}
\]

where \( v \) is the linear velocity vector and \( \omega \) is the angular rate, both expressed in \( \mathcal{B} \), and \( J \in \mathbb{R}^{3x3} \) is the moment of inertia tensor given by:

\[
J = I + mS(r)^T S(r)
\]

where:

\[
I = \begin{bmatrix}
    I_{xx} & I_{xy} & I_{xz} \\
    I_{xy} & I_{yy} & I_{yz} \\
    I_{xz} & I_{yz} & I_{zz}
\end{bmatrix}
= \int_V \begin{bmatrix}
    z_C^2 + y_C^2 & -x_C y_C & -x_C z_C \\
    -x_C y_C & x_C^2 + z_C^2 & -z_C y_C \\
    -x_C z_C & -z_C y_C & x_C^2 + y_C^2
\end{bmatrix} dx_C dy_C dz_C,
\]

and

\[
mS(r)^T S(r) = m \begin{bmatrix}
    r_x^2 + r_y^2 & -r_x r_y & -r_x r_z \\
    -r_x r_y & r_x^2 + r_z^2 & -r_z r_y \\
    -r_x r_z & -r_z r_y & r_x^2 + r_y^2
\end{bmatrix},
\]

with \( r \) the distance of displacement of the point \( p \) from the center of mass of the quadrotor.

The dynamic model of the quadrotor can be rewritten as:

\[
\begin{align*}
    \dot{\xi} &= v - R_\alpha \beta \omega \times r, \\
    m\dot{v} &= R_\alpha f, \\
    \dot{R} &= R_\alpha S(\beta \omega), \\
    J \dot{\omega} &= -\beta \omega \times J \beta \omega + \tau.
\end{align*}
\]

where the time derivative of the linear velocity (3.21) is given by:

\[
\dot{v} = R_\alpha \beta \omega \times (\beta \omega \times r) + R_\alpha \beta \omega \times r + \ddot{\xi}.
\]

The forces and torques actuating in the quadrotor, \( f, \tau \in \mathcal{B} \), consist of the aerodynamic force and moment vectors, gravitational forces, gyroscopic effects caused by the rotors, the torques generated by the four propellers, and the applied thrust. These
forces and torques can be expressed by

\[
\begin{align*}
R_f f &= -mg e_3 + R_{s_{e_3}} T + f_{\xi_d}, \\
\tau &= \tau_a + \tau_G - r \times (\hat{f} + R_{s_{e_3}} f_{\xi_d}),
\end{align*}
\]

(3.29)

where \(\tau_a\) is the applied torque vector given in the equation (3.3), \(e_3\) is the vector \([0\ 0\ 1]^T\) \((R_{s_{e_3}} \text{ results in the third column of the rotation matrix})\), \(\hat{f} = e_3 T\) is the translational control input \(T\) in the \(z\) axis, \(f_{\xi_d} = [a_x\ a_y\ a_z]^T\) is the aerodynamic force vector, whose components are in the \(x\), \(y\) and \(z\) axes respectively, and assumed external disturbances. \(\tau_G\) is the gyroscopic effect vector, and \(a_r = [a_p\ a_q\ a_r]^T\) is the aerodynamic moment vector expressed in the body-fixed frame. Given that the aerodynamic moments and the gyroscopic effects are assumed to be external disturbances, yields the vector \(\tau_d = a_r + \tau_G\).

By applying the equations of forces and torques (3.29) into the dynamic model (3.27), these equations of motion can be rewritten as follows:

\[
\begin{align*}
\dot{\xi} &= v - R_s \omega \times r, \\
\dot{v} &= -mg e_3 + R_{s_{e_3}} T + f_{\xi_d}, \\
\dot{R} &= R S(\omega), \\
J \omega &= -\omega \times J \omega + \tau_a + \tau_d - r \times (\hat{f} + R_{s_{e_3}} f_{\xi_d}).
\end{align*}
\]

(3.30)

By applying the linear acceleration (3.28) into (3.30), and after some algebraic manipulations, the following Newton-Euler equations of motion are obtained:

\[
\begin{align*}
R_{s_{e_3}} T + f_{\xi_d} &= m\ddot{\xi} - mR_s r \times \dot{\omega} - mR_s \omega \times (r \times \omega) + mg e_3, \\
\tau_a + \tau_d &= J \dot{\omega} - m r \times (r \times \dot{\omega}) + m \mathbf{r} \times R_s \dot{\xi} + \omega \times J \omega \\
&= -m \mathbf{r} \times (\omega \times (r \times \omega)) + m g r \times R_{s_{e_3}} e_3.
\end{align*}
\]

(3.31)

Considering the vector \(\mathbf{r}\) is null, the simplified model can be obtained from (3.31), which is given by

\[
\begin{align*}
\dot{\xi} &= v, \\
\dot{v} &= -mg e_3 + R_{s_{e_3}} T + f_{\xi_d}, \\
\dot{R} &= R S(\omega), \\
J \omega &= -\omega \times J \omega + \tau_a + \tau_d.
\end{align*}
\]

(3.32)

Whereas, besides the external disturbances, the system (3.32) should be exposed to
unmodeled dynamics and parametric uncertainties, it is rewritten as

\[
\dot{\xi} = v_x, 
\]

\[m \dot{v}_x = -mg\epsilon_3 + R_{p3} T + b, \]

\[\dot{R}_p = R_p S(\omega), \]

\[I^g \omega_{g,3} = -\omega_{g,3} \times I^g \omega_{g,3} + \tau_a + \tau_d. \]

(3.33a)

(3.33b)

(3.33c)

(3.33d)

where \( b = f + \delta \) is an unknown constant bias that represents the sum of the constant external disturbances, unmodeled dynamics and parametric uncertainties.

### 3.5 Chapter Conclusions

In this chapter, the dynamic modeling of the quadrotor UAV has been presented. The equations of motion have been described through Newton-Euler formulation for control design purposes.

The state-space representation of the 6DOF of the system were chosen as:

- \( \xi = [x, y, z]^T \) : The vector of the quadrotor’s position with respect to the inertial reference frame;

- \( \eta = [\phi, \theta, \psi]^T \) : Euler angles parametrizing the quadrotor’s orientation with respect to the inertial reference frame.

The control inputs were chosen the total thrust and the vector of the torques, represented by:

- \( u = [T, \tau_\phi, \tau_\theta, \tau_\psi]^T \).
In this chapter, it is addressed the quadrotor’s trajectory tracking problem. This problem is solved through the backstepping control approach with integral action, including rotation matrix properties in the control design. Two different strategies are designed and compared.

4.1 Backstepping path tracking with integral action

This section presents the design of a backstepping controller with integral action in order to solve the path tracking problem for the quadrotor UAV. The proposed control strategy extends the backstepping approach presented in Mahony and Hamel (2004) and Raptis et al. (2011), formulated for standard helicopters, to the quadrotor UAV, furthermore, including the integral action on the velocity errors of the regulated variables. Therefore, the objective is to design a feedback controller that manipulates the control inputs \((T, τ_a)\) of the system (3.33) in order to follow the bounded desired trajectories \(ξ_r(t) = [x_r(t) \ y_r(t) \ z_r(t)]^T\) and \(ψ_r(t)\), with null error for time-varying trajectories and constant disturbances. The backstepping controller is formulated based on the Newton-Euler model as follows.

- First Step

It is assumed the error associated to the translational position, which is the difference between actual and reference position, and is given by

\[ ε_1 = ξ(t) - ξ_r(t). \] (4.1)
The dynamics of this error are obtained as

$$\dot{\mathcal{E}}_1 = \dot{\xi}(t) - \dot{\xi}_r(t). \quad (4.2)$$

From this point on, the time dependence \((t)\) will be suppressed.

According to the backstepping standard approach described in Section 2.1, it is aimed to design a state feedback control law that can stabilize the system \((4.2)\). In this step, it is chosen \(\dot{\xi}\) as the state to be the virtual control input of the system \((4.2)\), for which \((\dot{\xi})_d = \phi_1(\xi)\), where \(d\) means the desired behavior of \(\dot{\xi}\). Rewritten \((4.2)\) yields

$$\dot{\mathcal{E}}_1 = \phi_1(\xi) - \dot{\xi}_r. \quad (4.3)$$

To guarantee the stability of the system, the following control Lyapunov function is proposed:

$$V_1(\xi) = \frac{1}{2} \mathcal{E}_1^T \mathcal{E}_1, \quad (4.4)$$

and its time derivative is given by

$$\dot{V}_1(\xi) = \mathcal{E}_1^T (\dot{\xi} - \dot{\xi}_r), \quad (4.5a)$$

$$= \mathcal{E}_1^T (\phi_1(\xi) - \dot{\xi}_r). \quad (4.5b)$$

Choosing the virtual feedback control law \(\phi_1(\xi)\) as follows

$$\phi_1(\xi) = \dot{\xi}_r - k_1 \mathcal{E}_1, \quad (4.6)$$

where \(k_1 > 0\), a constant positive diagonal matrix, and substituting \((4.6)\) in \((4.5b)\), yields to

$$\dot{V}_1(\xi) = \mathcal{E}_1^T (-k_1 \mathcal{E}_1) = -\mathcal{E}_1^T k_1 \mathcal{E}_1 < 0, \quad (4.7)$$

which is negative definite. An equivalent system of \((4.2)\) can be obtained by adding and subtracting \(\phi_1(\xi)\):

$$\dot{\mathcal{E}}_1 = -\dot{\xi}_r + \phi_1(\xi) + [\dot{\xi} - \phi_1(\xi)] = -k_1 \mathcal{E}_1 + [\dot{\xi} - \phi_1(\xi)]. \quad (4.8)$$

If it is possible to guarantee that the term \([\dot{\xi} - \phi_1(\xi)]\) converges to zero, the system \((4.2)\) will be asymptotically stable. Thus, by using the change of variable \(z_1 = [\dot{\xi} - \phi_1(\xi)]\),
and equations (3.33b) and (4.2), the translational error dynamics are rewritten as follows:

\[
\begin{align*}
\dot{E}_1 &= -k_1 E_1 + z_1, \\
\dot{z}_1 &= \dot{\theta}_p - \ddot{\xi}_r + k_1 (-k_1 E_1 + z_1).
\end{align*}
\] (4.9)

**Second Step**

Evaluating the error associated to \(\dot{\xi} - \phi_1(\xi)\), it is being assessed the translational velocity error, which is the difference between the real and the desired velocities. This step is developed in order to stabilize \(z_1\) at the origin and, consequently, achieve the translational position error convergence.

Moreover, at this step it is implemented the integral term \(X_\xi = \int_0^t z_1(\tau)d\tau\). As mentioned at the introduction, the integral action in the second step of the backstepping approach guarantees convergence for both constant and time-varying reference signals.

By using equation (3.33b) and adding the integral action, system (4.9) can be rewritten as

\[
\begin{align*}
\dot{E}_1 &= -k_1 E_1 + z_1, \\
\dot{X}_\xi &= z_1, \\
\dot{z}_1 &= -ge_3 + \frac{1}{m} R e_3 T + \frac{b}{m} - \ddot{\xi}_r + k_1 (-k_1 E_1 + z_1).
\end{align*}
\] (4.10)

At this step, the virtual control input is chosen as the term that contains the rotation matrix and the total thrust \((R e_3 T)_d = \phi_2(R, T)\). Thus, the second control Lyapunov function is chosen as

\[
V_2(E_1, z_1, X_\xi) = V_1 + \frac{1}{2} z_1^T z_1 + \frac{1}{2} X_\xi^T k_{X_\xi} X_\xi,
\] (4.11)

where \(k_{X_\xi} = k_{X_\xi}^T > 0\), a constant positive diagonal matrix. The time derivative of \(V_2(E_1, z_1, X_\xi)\) is given by

\[
\begin{align*}
\dot{V}_2(E_1, z_1, X_\xi) &= \dot{E}_1^T (-k_1 E_1 + z_1) + z_1^T \dot{z}_1 + X_\xi^T k_{X_\xi} \dot{X}_\xi \\
&= \dot{E}_1^T (-k_1 E_1 + z_1) + z_1^T \left(-ge_3 + \frac{1}{m} \phi_2 + \frac{b}{m} - \ddot{\xi}_r + k_1 (-k_1 E_1 + z_1)\right) \\
&\quad + X_\xi^T k_{X_\xi} z_1.
\end{align*}
\] (4.12)

The virtual control law \(\phi_2(R, T)\) is designed as

\[
\phi_2(R, T) = m(ge_3 + \ddot{\xi}_r - k_1 (-k_1 E_1 + z_1)) - E_1 - k_2 z_1 - k_{X_\xi} X_\xi,
\] (4.13)

where \(k_2 = k_2^T > 0\).
Replacing this control law in (4.12), yields

\[
\dot{V}_2(\mathcal{E}_1, z_1, X_\xi) = -\mathcal{E}_1^T k_1 \mathcal{E}_1 - z_1^T k_2 z_1 + z_1^T \frac{b}{m},
\]

(4.14)

what shows that the Lyapunov function candidate \( V_2 \) has convergence for the case that the bias term \( b = 0 \). Nevertheless, \( V_2 \) would have asymptotically convergence for the states \( \xi \) and \( z_1 \), but without guarantee about \( X_\xi \).

The system (4.10) can be rewritten by adding and subtracting \( \frac{1}{m} \phi_2 \) as follows

\[
\dot{\mathcal{E}}_1 = - k_1 \mathcal{E}_1 + z_1,
\]

\[
\dot{X}_\xi = z_1,
\]

\[
\dot{z}_1 = - \mathcal{E}_1 - k_2 z_1 - k_{X\xi} X_\xi + \frac{b}{m} + \frac{1}{m} [\text{Re}_3 T - \phi_2],
\]

(4.15)

from which the change of variable results in \( z_2 = [\text{Re}_3 T - \phi_2] \).

The asymptotically stability analysis of the system (4.15) will be completed via Matrosov’s Theorem.

**Theorem 2.** Under the assumption that \( \lim_{t \to \infty} z_2 = 0 \), the closed loop of (4.15) is UGAS in the equilibrium point \( (\mathcal{E}_1, z_1, X_\xi) = (0, 0, k_{X\xi} \frac{z_2}{m}) \) for any constant bias \( b \).

**Proof.** Stability analysis via Matrosov’s Theorem:

Stability of (4.15) will be evaluated according to Theorem 1, as stated in Section 2.2. Firstly, it is assumed convergence of the term \( z_2 \), that will be assessed in the next step of the backstepping procedure. Thus, system (4.15) is rewritten as

\[
\dot{\mathcal{E}}_1 = - k_1 \mathcal{E}_1 + z_1,
\]

\[
\dot{X}_\xi = z_1,
\]

\[
\dot{z}_1 = - \mathcal{E}_1 - k_2 z_1 - k_{X\xi} X_\xi + \frac{b}{m}. \tag{4.16}
\]

Therefor, the system (4.16) is UGS in the origin \( (\mathcal{E}_1, z_1, X_\xi) = (0, 0, 0) \) for \( b = 0 \), what means that the Assumption 1 is met.

The following functions are defined:

\[
\varphi(\xi, \xi_r(t)) := \mathcal{E}_1, \tag{4.17}
\]

\[
W_1(\mathcal{E}_1, z_1, X_\xi) := V_2, \tag{4.18}
\]

\[
W_2(z_1, X_\xi) := X_\xi^T z_1, \tag{4.19}
\]

\[
Y_1(\mathcal{E}_1, z_1, X_\xi) := -\mathcal{E}_1^T k_1 \mathcal{E}_1 - z_1^T k_2 z_1, \tag{4.20}
\]

\[
Y_2(\mathcal{E}_1, z_1, X_\xi) := z_1^T z_1 - X_\xi^T \mathcal{E}_1 - X_\xi^T k_2 z_1 - X_\xi^T k_{X\xi} X_\xi. \tag{4.21}
\]

Thus, due to the boundedness of \( \xi_r(t) \) and the continuity of \( \xi(t) \), for bounded \( (\mathcal{E}_1, z_1, X_\xi) \),
the functions $\varphi(\xi, \xi_r(t))$, $W_1(\xi_1, z_1, \chi_\xi)$ and $W_2(z_1, \chi_\xi)$ are bounded, what meets to the Assumption 2. Moreover, if $Y_1(\xi_1, z_1, \chi_\xi) = 0$, implies $Y_2(\xi_1, z_1, \chi_\xi) = -\chi_\xi^T k_{\chi_\xi} \chi_\xi \leq 0$, according to Assumption 3. Furthermore, in case of $Y_1(\xi_1, z_1, \chi_\xi) = Y_2(\xi_1, z_1, \chi_\xi) = 0$, implies the states $(\xi_1, z_1, \chi_\xi) = 0$, in conformity to Assumption 4. What proves that the origin $(\xi_1, z_1, \chi_\xi) = 0$ is UGAS.

Assessing the case that the bias $b \neq 0$, if $-k_{\chi_\xi} \chi_\xi + \frac{b}{m} = 0$, the system (4.16) will have a new constant equilibrium point at $(\xi_1, z_1, \chi_\xi) = (0, 0, k^{-1}_{\chi_\xi} \frac{b}{m}).$

Defining $\breve{\chi}_\xi = \chi_\xi - k_{\chi_\xi} \frac{b}{m}$, system (4.16) can be rewritten as

$$
\begin{align*}
\dot{\breve{\xi}}_1 &= -k_1 \xi_1 + z_1, \\
\dot{\breve{\chi}}_\xi &= z_1, \\
\dot{z}_1 &= -\xi_1 - k_2 z_1 - k_{\chi_\xi}\breve{\chi}_\xi, \\
\end{align*}

(4.22)
$$

which is the same as (4.16) for $b = 0$, what gives that (4.16) is UGAS for any constant bias $b$, furthermore, the system (4.15) is UGAS for any constant bias $b$ and $z_2 = 0$.

\[ \Box \]

- Third Step

When the term $z_2$ converges to zero, the system (4.15) is asymptotically stable. Therefore, at the third step it must be guaranteed that the difference between the real vector $Re_3 T$ and the desired one, $\phi_3 = (Re_3 T)_d$, converges to zero. Hence, the time-derivative of $z_2$ is computed as

$$
\begin{align*}
\dot{z}_2 &= dRe_3 T \frac{dt}{dt} - d\phi_3 \frac{dt}{dt} \\
&= Re_3 \ddot{T} + RS(w)e_3 T - m \ddot{\xi}_r + mk_{\xi_1} \dot{\xi}_1 + m\dot{\xi}_1 + mk_{\chi_\xi} \dot{z}_1 + mk_{\chi_\xi} \ddot{z}_1. \\
\end{align*}

(4.23)
$$

At this step the virtual control input is chosen as the term $\phi_3(R, \dot{T}, \dot{\omega}) = (Re_3 \ddot{T} + RS(\omega)e_3 T)_d$. The third control Lyapunov function is chosen as

$$
V_3(\xi_1, z_1, \chi_\xi, z_2) = V_2 + \frac{1}{2} z_2^T z_2,
$$

(4.24)

and its time derivative is given by

$$
\begin{align*}
\dot{V}_3(\xi_1, z_1, \chi_\xi, z_2) &= \xi_1(-k_1 \xi_1 + z_1) + z_1^T \left(-\xi_1 - k_2 z_1 - k_{\chi_\xi} \chi_\xi + \frac{b}{m} + \frac{1}{m} z_2 \right) + \chi_\xi^T k_{\chi_\xi} \ddot{z}_1 \\
&\quad + z_2^T \left(\phi_3 - m \ddot{\xi}_r + mk_{\xi_1} \dot{\xi}_1 + m(-k_1 \xi_1 + z_1) + mk_{\chi_\xi} \dot{z}_1 + mk_{\chi_\xi} \ddot{z}_1 \right). \\
\end{align*}

(4.25)
$$

Accordingly, now the virtual control law $\phi_3$ is shaped as

$$
\phi_3 = m \ddot{\xi}_r - mk_{\xi_1} \dot{\xi}_1 - m(-k_1 \xi_1 + z_1) - mk_{\chi_\xi} \dot{z}_1 - mk_{\chi_\xi} \ddot{z}_1 - \frac{1}{m} \dot{z}_1 - k_3 \ddot{z}_2.
$$

(4.26)
Therefore, applying the virtual control law (4.26) in (4.25), results in

\[
\dot{V}_3(\mathcal{E}_1, z_1, \chi, z_2) = -(\mathcal{E}_1^T k_1 \mathcal{E}_1 - z_1^T k_2 z_1 - z_2^T k_3 z_2 + z_1^T \frac{b}{m} < 0, \tag{4.27}
\]

which is asymptotically stable, given that \(z_1^T \frac{b}{m}\) vanishes, resulted from the integral action \(\chi\), as proved in Theorem 2.

By adding and subtracting \(\phi_3\) from (4.23) the equivalent system is obtained:

\[
\dot{z}_2 = -\frac{1}{m} z_1 - k_3 z_2 + \left[\text{Re} \ 3 \dot{T} + \text{RS}(w)e_3 T - \phi_3\right], \tag{4.28}
\]

in which the change of variable \(z_3 = [\text{Re} \ 3 \dot{T} + \text{RS}(w)e_3 T - \phi_3]\) is used.

- Fourth Step

The difference between \((\text{Re} \ 3 \dot{T} + \text{RS}(w)e_3 T) - (\text{Re} \ 3 \dot{T} + \text{RS}(w)e_3 T)_d\) must be null in order to achieve convergence of equation (4.28), which is guaranteed in this step considering

\[
\dot{z}_3 = d \frac{\text{Re} \ 3 \dot{T} + \text{RS}(w)e_3 T}{dt} - d \frac{\phi_3}{dt} = \text{Re} \ 3 \dot{T} - \text{RS}(e_3)\dot{\omega} T + \text{RS}(\omega)S(\omega)e_3 T + 2\text{RS}(\omega)e_4 \dot{T} - m \dddot{\xi}_r + mk_1 \dddot{\xi}_1
\]

\[
+ m \dddot{\xi}_1 + mk_2 \dddot{z}_1 + mk_3 \dddot{z}_2 + \frac{1}{m} \dot{z}_1 + k_3 \dot{z}_2. \tag{4.29}
\]

Considering the following operations

\[
\text{Re} \ 3 \dot{T} = R \begin{bmatrix} 0 \\ 0 \\ \dot{T} \end{bmatrix} = R \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}, \tag{4.30}
\]

\[
\text{RS}(e_3)\dot{\omega} T = R \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}, \tag{4.31}
\]

the term \((\text{Re} \ 3 \dot{T} - \text{RS}(e_3)\dot{\omega} T)\) of equation (4.29), using \(\dot{\omega} = [\dot{p} \ \dot{q} \ \dot{r}]^T\) and properties of skew-symmetric matrices (Spong et al., 2006), can be reorganized as follows:

\[
\text{Re} \ 3 \dot{T} - \text{RS}(e_3)\dot{\omega} T = R \begin{bmatrix} 0 & T & 0 \\ -T & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix}. \tag{4.32}
\]
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Therefore, equation (4.29) can be rewritten as

\[
\dot{z}_3 = \begin{bmatrix} 0 & T & 0 \\ -T & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{T} \end{bmatrix} + 2RS(\omega)e_3\dot{T} + RS(\omega)S(\omega)e_3T - m\dddot{\xi} + mk_1\ddot{\xi}_1 \\
+m\ddot{\xi}_1 + mk_2\ddot{z}_1 + mk_{x\xi}\ddot{z}_1 + \frac{1}{m}\dot{z}_1 + k_3\dot{z}_2.
\]

(4.33)

At this stage, the vector \( \phi_4 = (\dot{p} \quad \dot{q} \quad \dot{T}) \) is the virtual control input, and the control Lyapunov function is chosen as

\[
V_4(\xi_1, z_1, \xi_2, z_2, z_3) = V_3 + \frac{1}{2}z_3^Tz_3,
\]

(4.34)

where its time derivative is given by

\[
\dot{V}_4(\xi_1, z_1, \xi_2, z_2, z_3) = \dot{E}_1(-k_1\dot{E}_1 + \dot{z}_1) + \ddot{z}_1^T \left( -\dot{E}_1 - k_2\ddot{z}_1 - k_{x\xi}\dot{\xi}_1 + \frac{b}{m} + \frac{1}{m}\dot{z}_2 \right) \\
+ \dot{\xi}_1^T k_{x\xi}\ddot{z}_1 + \ddot{z}_2^T \left( -\frac{1}{m}\dot{z}_1 - k_3\ddot{z}_2 + \ddot{z}_3 \right) + \ddot{z}_3^T \left( R -T 0 0 0 \right) \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{T} \end{bmatrix} \\
+ 2RS(\omega)e_3\dot{T} + RS(\omega)S(\omega)e_3T - m\dddot{\xi} + mk_1\ddot{\xi}_1 + m\ddot{\xi}_1 \\
+ mk_2\ddot{z}_1 + mk_{x\xi}\ddot{z}_1 + \frac{1}{m}\dot{z}_1 + k_3\dot{z}_2.
\]

(4.35)

Considering the rotation matrix properties (Spong et al., 2006), the virtual control law is defined as

\[
\phi_4 = \begin{bmatrix} 0 & -\frac{1}{T} & 0 \\ \frac{1}{T} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \left( -2RS(\omega)e_3\dot{T} - RS(\omega)S(\omega)e_3T + m\dddot{\xi} - mk_1\ddot{\xi}_1 - m\ddot{\xi}_1 \\
- mk_2\ddot{z}_1 - mk_{x\xi}\ddot{z}_1 - \frac{1}{m}\dot{z}_1 - k_3\ddot{z}_2 - \ddot{z}_2 - k_4\ddot{z}_3 \right), \quad k_4 = k_4^T > 0.
\]

(4.36)

Thereby, the time derivative of the candidate Lyapunov \( V_4 \) becomes negative definite:

\[
\dot{V}_4(\xi_1, z_1, \xi_2, z_2, z_3) = -\dot{E}_1^T k_1\dot{E}_1 - z_2^T k_2z_1 - z_2^T k_3z_2 - z_3^T k_4z_3 + \frac{z_3^T b}{m} < 0.
\]

(4.37)

By applying (4.36) into (4.33), the closed-loop dynamics of \( z_3 \) is given by

\[
\dot{z}_3 = -z_2 - k_4z_3,
\]

(4.38)

which is asymptotically stable, assuming that the term \( z_1^T b \) vanishes based on Theorem 2.

- Fifth Step
Comparing the equivalent equations (4.29) and (4.33), it can be observed that the third element of the angular acceleration vector, $\omega_{\theta\theta,\tau}$, is not necessary to the control design in the four steps of the backstepping developed until now. This fact allows yaw control design in order to fit the complete control system (Mahony and Hamel, 2004).

At this step, it is considered the error between real and reference values of yaw angle, $\psi$

$$\mathcal{E}_\psi = \psi - \psi_r,$$  \hspace{1cm} (4.39)

and its time derivative is given by

$$\dot{\mathcal{E}}_\psi = \dot{\psi} - \dot{\psi}_r.$$  \hspace{1cm} (4.40)

The virtual control input is chosen as $(\dot{\psi})_d = \phi_5(\mathcal{E}_\psi)$, and the candidate Lyapunov function is selected as

$$V_5(\mathcal{E}_\psi) = \frac{1}{2} \mathcal{E}_\psi^T \mathcal{E}_\psi.$$  \hspace{1cm} (4.41)

Its time derivative is given by

$$\dot{V}_5(\mathcal{E}_\psi) = \mathcal{E}_\psi^T (\phi_5 - \dot{\psi}_r).$$  \hspace{1cm} (4.42)

By choosing the virtual control law as

$$\phi_5(\mathcal{E}_\psi) = \dot{\psi}_r - k_5 \mathcal{E}_\psi, \quad k_5 = k_5^T > 0,$$  \hspace{1cm} (4.43)

the time derivative of the candidate Lyapunov function becomes negative definite as follows

$$\dot{V}_5(\mathcal{E}_\psi) = \mathcal{E}_\psi^T (-k_5 \mathcal{E}_\psi) = -\mathcal{E}_\psi^T k_5 \mathcal{E}_\psi < 0,$$  \hspace{1cm} (4.44)

and so, the equation (4.40), by adding and subtracting $\phi_5$, becomes

$$\dot{\mathcal{E}}_\psi = -\dot{\psi}_r + \phi_5 + [\dot{\psi} - \phi_5].$$  \hspace{1cm} (4.45)

Now, the change of variable $z_4 = [\dot{\psi} - \phi_5]$ is used.

**Sixth Step**

At this step, the error associated to the time derivative of the yaw angle is stabilized,
4.1. BACKSTEPPING PATH TRACKING WITH INTEGRAL ACTION

in which the integral term \( \mathcal{X}_\psi = \int_0^t z_4(\tau) d\tau \) is added. Thus, considering the system

\[
\begin{align*}
\dot{\psi} &= -k_5 \psi + z_4, \\
\dot{X}_\psi &= z_4, \\
\dot{z}_4 &= \ddot{\psi} - \ddot{\psi}_r + k_5 \ddot{\psi},
\end{align*}
\]

the virtual control input is chosen as the desired yaw angle acceleration \( \phi_6 = (\ddot{\psi})_d \).

The following control Lyapunov function is proposed

\[
V_6(\mathcal{E}_\psi, \mathcal{X}_\psi, z_4) = V_5 + \frac{1}{2} z_4^T z_4 + \frac{1}{2} \mathcal{X}_\psi^T k \mathcal{X}_\psi,
\]

and its time derivative is given by

\[
\dot{V}_6(\mathcal{E}_\psi, \mathcal{X}_\psi, z_4) = \mathcal{E}_\psi^T (-k_5 \mathcal{E}_\psi + z_4) + z_4^T (\phi_6 - \ddot{\psi}_r + k_5 \ddot{\psi}) + \mathcal{X}_\psi^T k \mathcal{X}_\psi z_4, \quad k \mathcal{X}_\psi > 0. \tag{4.48}
\]

Thus, the virtual control law, \( \phi_6 \), is designed as follows

\[
\phi_6 = \ddot{\psi}_r - k_5 \ddot{\psi} - k_6 z_4 - \mathcal{E}_\psi - k \mathcal{X}_\psi.
\]

Applying (4.49) in (4.48), results in

\[
\dot{V}_6(\mathcal{E}_\psi, \mathcal{X}_\psi, z_4) = -\mathcal{E}_\psi^T k_5 \mathcal{E}_\psi - z_4^T k_6 z_4 \leq 0, \quad k_5, k_6 > 0, \tag{4.50}
\]

which is semi-definite negative, since \( \mathcal{X}_\psi \) is not explicit in the time derivative of this candidate Lyapunov function. However, analyzing by Theorem 2, this system is similar to (4.14), what implies that the origin \( (\mathcal{E}_\psi, \mathcal{X}_\psi, z_4) = (0, 0, 0) \) is UGAS.

Now, in order to obtain the closed-loop dynamics of the quadrotor UAV with the proposed backstepping controller, firstly, the desired yaw angle acceleration, \( \phi_6 = (\ddot{\psi})_d \), is mapped to the angular acceleration \( \dot{r}_d \). Hence, assuring that pitch and roll angles belong to the interval \( (-\pi/2, \pi/2) \), and from (3.18), the following second-order differential kinematics is obtained

\[
\ddot{\psi} = -W \eta^{-1} W \eta^{-1} \omega + W \eta^{-1} \dot{\omega}. \tag{4.51}
\]

By multiplying both sides of (4.51) by \( e_3^T = [0 \ 0 \ 1] \), and using \( \dot{\omega} = \begin{bmatrix} \dot{p} & \dot{q} & \dot{r} \end{bmatrix}^T \), the following equation is obtained

\[
\ddot{\psi} = -[0 \ 0 \ 1] W \eta^{-1} W \eta^{-1} \omega + \dot{q} \frac{sin(\phi)}{cos(\theta)} + \dot{r} \frac{cos(\phi)}{cos(\theta)}. \tag{4.52}
\]
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Assuming (4.52) equals to \((\dot{\psi})_d\) and replacing it by \(\phi_\ell\) in (4.49), yields to

\[
\dot{r}_d = \frac{\cos(\theta)}{\cos(\phi)} \left( \dot{\psi}_r - k_5 \dot{\psi} - k_6 z_4 - \mathcal{E}_\phi - k_{\chi\phi} \chi_\phi + [0 \ 0 \ 1] W \eta^{-1} \dot{W} \eta^{-1} \omega - q_d \frac{\sin(\phi)}{\cos(\theta)} \right).
\] (4.53)

Therefore, the complete dynamics of the quadrotor UAV in closed-loop can be rewritten through equations (4.15), (4.28), (4.33) and (4.46):

\[
\begin{align*}
\dot{E}_1 &= -k_1 E_1 + z_1, \\
\dot{X}_\psi &= z_4, \\
\dot{z}_1 &= -E_1 - k_2 z_1 - k_{\chi\xi} X_\xi + \frac{1}{m} z_2 + \frac{b}{m}, \\
\dot{z}_2 &= -\frac{1}{m} z_1 - k_3 z_2 + z_3, \\
\dot{z}_3 &= \begin{bmatrix} 0 & T & 0 \\ -T & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{T} \end{bmatrix}_d + 2RS(\omega)e_3 \dot{T} + RS(\omega)S(\omega)e_3 T - m \ddot{\xi}_r + mk_1 \dddot{E}_1 \\
&\quad + m \dddot{X}_\xi + mk_2 \dddot{z}_1 + \frac{1}{m} \dddot{z}_1 + k_3 \dddot{z}_2 \\
\dot{E}_\psi &= -k_5 E_\psi + z_4, \\
\dot{X}_\psi &= z_4, \\
\dot{z}_4 &= -[0 \ 0 \ 1] W \eta^{-1} \dot{W} \eta^{-1} \omega + q \frac{\sin(\phi)}{\cos(\theta)} + \dot{r}_d \frac{\cos(\phi)}{\cos(\theta)} - \dddot{\psi}_r + k_5 \dddot{E}_\phi,
\end{align*}
\] (4.54)

with the feedback control inputs (4.36) and (4.53) designed to follow the desired trajectories.

Using these control inputs, the applied control signals to the system (3.33) are obtained as follows

\[
\tau_a = I \ddot{\omega} + \omega \times I \omega, \\
T = \int \int \dot{T}_d.
\] (4.55) (4.56)

In order to evaluate the complete stability of the quadrotor UAV’s flight, it is analyzed the sum of all control Lyapunov functions chosen in the backstepping procedure:

\[
V = \frac{1}{2} E_1^T E_1 + \frac{1}{2} z_1^T z_1 + \frac{1}{2} X_\xi^T k_{\chi\xi} X_\xi + \frac{1}{2} z_2^T z_2 + \frac{1}{2} z_3^T z_3 + \frac{1}{2} E_\psi^T E_\psi + \frac{1}{2} z_4^T z_4 + \frac{1}{2} X_\phi^T k_{\chi\phi} X_\phi.
\] (4.57)
and its time derivative is given by
\begin{align*}
\dot{V} = & \mathcal{E}_1(-k_1 \mathcal{E}_1 + z_1) + z_1^T \left( -\mathcal{E}_1 - k_2 z_1 + \frac{1}{m} z_2 - k_{x\xi} \mathcal{X}_z \right) + \mathcal{X}_z^T k_{x\xi} z_1 + z_2^T \left( -\frac{1}{m} z_1 - k_3 z_2 + z_3 \right) \\
& + z_3^T \left[ R \begin{bmatrix} 0 & T & 0 \\ -T & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{\mathcal{E}}_1 \end{bmatrix} \right] + 2RS(\omega)e_3 \dot{T} + RS(\omega)S(\omega)e_3 T - m \ddot{\mathcal{E}}_1^T + m \ddot{\mathcal{E}}_1 + m \ddot{\mathcal{E}}_1 + m k_2 z_1 + m k_{x\xi} \dot{z}_1 + \frac{1}{m} \dot{z}_1 + k_3 \dot{z}_2 \\
& + \mathcal{E}_\psi(-k_3 \mathcal{E}_\psi + z_4) + z_4(\ddot{\psi} - \psi_r + k_5 \dot{\psi}) + \mathcal{X}_\psi k_{x\psi} z_4.
\end{align*}

(4.58)

By applying the control inputs (4.36) and (4.53), yields to:
\begin{equation}
\dot{V} = -\mathcal{E}_1^T k_1 \mathcal{E}_1 - z_1^T k_2 z_1 - z_2^T k_3 z_2 - z_3^T k_4 z_3 - k_5 \mathcal{E}_\psi^T \mathcal{E}_\psi - k_6 z_4^T z_4 + z_1 \frac{b}{m}. \tag{4.59}
\end{equation}

Assuming that the term \( z_1^T \frac{b}{m} \) vanishes based on Theorem 2, (4.59) is negative definite, ensuring that the origin of the closed-loop system (4.54) with (4.36) and (4.53) is asymptotically stable.

The stability analyses of the complete system is given by the sum of the six Lyapunov functions obtained in the control design
\begin{equation}
V = \frac{1}{2} \mathcal{E}_1^T \mathcal{E}_1 + \frac{1}{2} z_1^T z_1 + \frac{1}{2} \mathcal{X}_z^T k_{x\xi} \mathcal{X}_z + \frac{1}{2} z_2^T z_2 + \frac{1}{2} z_3^T z_3 + \frac{1}{2} \mathcal{E}_\psi^T \mathcal{E}_\psi + \frac{1}{2} z_4^T z_4 + \frac{1}{2} \mathcal{X}_\psi k_{x\psi} \mathcal{X}_\psi, \tag{4.60}
\end{equation}

and its time derivative is given by
\begin{align*}
\dot{V} = & \mathcal{E}_1(-k_1 \mathcal{E}_1 + z_1) + z_1^T \left( -\mathcal{E}_1 - k_2 z_1 + \frac{1}{m} z_2 - k_{x\xi} \mathcal{X}_z \right) + \mathcal{X}_z^T k_{x\xi} z_1 \\
& + z_3^T \left( \frac{1}{m} z_1 - k_3 z_2 + z_3 \right) + z_3^T \left[ R \begin{bmatrix} 0 & T & 0 \\ -T & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{\mathcal{E}}_1 \end{bmatrix} \right] + 2RS(\omega)e_3 \dot{T} \\
& + RS(\omega)S(\omega)e_3 T - m \ddot{\mathcal{E}}_1^T + m \ddot{\mathcal{E}}_1 + m k_2 \dot{z}_1 + m k_{x\xi} \dot{z}_1 + \frac{1}{m} \dot{z}_1 + k_3 \dot{z}_2 \\
& + \mathcal{E}_\psi(-k_3 \mathcal{E}_\psi + z_4) + z_4(\ddot{\psi} - \psi_r + k_5 \dot{\psi}) + \mathcal{X}_\psi k_{x\psi} z_4.
\end{align*}

(4.61)

(4.62)

(4.63)

(4.64)

which is asymptotically stable.

### 4.1.1 Simulation Results

Simulations were carried out to observe and assess the performance of the proposed controller. It is used as reference trajectory an eight-shaped path in the \( \mathbb{R}^3 \) space described
by

\[
x_r = \frac{1}{2} \cos \left( \frac{\pi t}{40} \right) \text{m},
\]

\[
y_r = \frac{1}{2} \sin \left( \frac{\pi t}{20} \right) \text{m},
\]

\[
z_r = 2 - \frac{1}{2} \cos \left( \frac{\pi t}{40} \right) \text{m},
\]

with the quadrotor yaw angle turned dynamically to the desired position, defined by

\[
\psi_r = \arctan \left( \frac{\dot{y}_r}{\dot{x}_r} \right) \text{rad}.
\]

The quadrotor UAV’s parameters were obtained from Raffo (2011), and are given in table 4.1. The simulations were performed in Matlab/Simulink® software.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass of the quadrotor UAV</td>
<td>(m)</td>
<td>2.24 kg</td>
</tr>
<tr>
<td>Distance between the rotors and the vehicle’s center of gravity</td>
<td>(l)</td>
<td>0.332 m</td>
</tr>
<tr>
<td>Thrust coefficient of the rotors</td>
<td>(b)</td>
<td>9.5e − 6 Ns²</td>
</tr>
<tr>
<td>Drag coefficient of the rotors</td>
<td>(k_r)</td>
<td>1.7e − 7 Nms²</td>
</tr>
<tr>
<td>Gravitational acceleration</td>
<td>(g)</td>
<td>9.81 m/s²</td>
</tr>
<tr>
<td>Moment of inertia around the (x)-axis</td>
<td>(I_{xx})</td>
<td>0.0363 Kg.m²</td>
</tr>
<tr>
<td>Moment of inertia around the (y)-axis</td>
<td>(I_{yy})</td>
<td>0.0363 Kg.m²</td>
</tr>
<tr>
<td>Moment of inertia around the (z)-axis</td>
<td>(I_{zz})</td>
<td>0.0615 Kg.m²</td>
</tr>
<tr>
<td>Nominal force of each propeller</td>
<td>(f_i)</td>
<td>12.2 N</td>
</tr>
</tbody>
</table>

The initial conditions for this scenario are \(\xi_0 = [0.25 \ 0.25 \ 0.5]^T \text{m and } \eta_0 = [0 \ 0 \ \frac{\pi}{4}]^T \text{rad.}\) It was considered an initial thrust for hovering flight of \(T = gm \approx 21.97 \text{N.}\)

Using the parameters of 4.1 in equations (3.1), (3.3) and (3.4), it is obtained the maximum total thrust and torques possible to be applied to the quadrotor, given by \(u_{max} = [48.8 \text{ N} \ 4.05 \text{ N.m} \ 4.05 \text{ N.m} \ 0.437 \text{ N.m}]^T.\) In the simulations it is used saturation to limit the control input efforts, bounded by values presented in the table 4.2.

<table>
<thead>
<tr>
<th>Control input</th>
<th>Minimum value</th>
<th>Maximum value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T)</td>
<td>0.01 N</td>
<td>48.0 N</td>
</tr>
<tr>
<td>(\tau_{\phi a})</td>
<td>−4.0 N.m</td>
<td>+4.0 N.m</td>
</tr>
<tr>
<td>(\tau_{\theta a})</td>
<td>−4.0 N.m</td>
<td>+4.0 N.m</td>
</tr>
<tr>
<td>(\tau_{\psi a})</td>
<td>−0.425 N.m</td>
<td>+0.425 N.m</td>
</tr>
</tbody>
</table>
The occurrence of external disturbances is simulated as given in table 4.3. It was also considered in simulation the presence of ±30\% parametric uncertainties in the total mass and in the moments of inertia of the quadrotor UAV.

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Disturbance</th>
<th>Initial time</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>1 N</td>
<td>10 s</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0.4 N.m</td>
<td>20 s</td>
</tr>
<tr>
<td>y</td>
<td>1 N</td>
<td>30 s</td>
</tr>
<tr>
<td>(\theta)</td>
<td>0.4 N.m</td>
<td>40 s</td>
</tr>
<tr>
<td>z</td>
<td>1 N</td>
<td>50 s</td>
</tr>
<tr>
<td>(\psi)</td>
<td>0.15 N.m</td>
<td>60 s</td>
</tr>
</tbody>
</table>

Table 4.3: Simulation disturbances.

By applying the control laws in the system (3.33), this backstepping controller can be viewed as a nonlinear Proportional Integral Derivative with Feed Forward action (PID+FF) controller. In such a way that the matrices of gain \(k_1\), \(k_2\) and \(k_1\times\xi\) can be viewed as the proportional, derivative and integral gains, respectively. Furthermore, the gains \(k_3\), \(k_4\) can be considered proportional and derivative gains as a second loop of the same controller. For the yaw angle controller, the gains \(k_5\), \(k_6\) and \(k_{x\psi}\) can be viewed as the proportional, derivative and integral gains, respectively. The backstepping controller gains were tuned for the simulation without integral action as follows

\[
k_1 = \begin{bmatrix} 20.8 & 0 & 0 \\ 0 & 20.8 & 0 \\ 0 & 0 & 156 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 48 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 168 \end{bmatrix},
\]

\[
k_3 = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad k_4 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},
\]

\[
k_{x\xi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
k_5 = 20, \quad k_6 = 4, \quad k_{x\psi} = 0.
\]
The controller gains were tuned for the simulation with integral action as

\[
\begin{align*}
k_1 &= \begin{bmatrix}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 4.6
\end{bmatrix}, & k_2 &= \begin{bmatrix}
10 & 0 & 0 \\
0 & 10 & 0 \\
0 & 0 & 10
\end{bmatrix}, \\
k_3 &= \begin{bmatrix}
6 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & 6
\end{bmatrix}, & k_4 &= \begin{bmatrix}
20 & 0 & 0 \\
0 & 20 & 0 \\
0 & 0 & 20
\end{bmatrix}, \\
k_{x\xi} &= \begin{bmatrix}
10 & 0 & 0 \\
0 & 10 & 0 \\
0 & 0 & 10
\end{bmatrix}, \\
k_5 &= 20, & k_6 &= 12, & k_{x\psi} &= 20.
\end{align*}
\]

Two simulations were carried out in order to corroborate the proposed backstepping controller. The first one was obtained with the backstepping controller without integral action, which is being called standard backstepping controller, while the second one simulate the proposed backstepping control law with integral action. All the parameters, the trajectory reference and configuration were the same for both controllers, they differ only by the integral action’s gains that appear only in the proposed controller.

Figure 4.1 shows the trajectory tracking results in a three dimensional view for both controllers. The quadrotor UAV starts displaced from the desired trajectory and track it. As proposed, when subjected to parametric uncertainties the backstepping controller with integral action achieved null steady-state error, while the one without integral action presented offset in steady-state.

Figure 4.2 shows the trajectory tracking results separated by axis, including the yaw angle following. This angle, as well as the other variables, presents convergence for both controllers.

The presence of external disturbance increase trajectory tracking errors, as showed in Figure 4.3, in which it can be observed that the controller with integral action provides attenuation of the errors, as expected by the control strategy design. Furthermore, it is observed that the integral action reduces the rise time and increases the overshoot, also expected in the control design. The controller without integral action keeps a trajectory tracking when submitted to theses disturbances, although without convergence, having an offset to the trajectory.

Figure 4.4 shows the quadrotor UAV orientation. As proposed, the controller maintained roll and pitch angles stabilized. It is not necessary that these angles converge to zero, it can be observed in simulation that they tilt the vehicle to compensate the occurrence of disturbances (lateral and longitudinal forces). For disturbance as torques in the quadrotor UAV’s axes, counter torque is applied by the controller to keep the vehicle in the desired
4.1. BACKSTEPPING PATH TRACKING WITH INTEGRAL ACTION

The control input signals’ behavior are shown in the figure 4.5. It can be observed that the control with integral action is more aggressive, as expected.

The saturation in the control input efforts is observed in this simulation, more clearly the presence of high error values and parameters uncertainty, as highlighted in the first
three seconds of simulation showed on figure 4.6. Despite of the boundedness of the input signals, the stabilization is preserved.
4.1. BACKSTEPPING PATH TRACKING WITH INTEGRAL ACTION

(a) Standard Backstepping

(b) Backstepping with Integral Action

Figure 4.3: Trajectories Errors
Figure 4.4: Quadrotor UAV Orientation
4.1. BACKSTEPPING PATH TRACKING WITH INTEGRAL ACTION

Figure 4.5: Control Input Signals
CHAPTER 4. CONTROL STRATEGIES

Figure 4.6: Control Input Signals - Saturation

(a) Standard Backstepping

(b) Backstepping with Integral Action
4.2 Backstepping Attitude Controller for the Quadrotor UAV

In this section the backstepping controller for quadrotor path tracking using rotation matrix in the attitude stabilization problem is developed with the objective of solving the path tracking problem for the quadrotor UAV, avoiding singularity points. In the first controller, designed in Section 4.1, the third column of the rotation matrix was used as a virtual control input, differently from the next controller, in which the complete rotation matrix is used in the control design.

As commented in Tayebi and McGilvray (2006), attitude controller is an essential component in aerial vehicles control, in order to provide the desired and also correct orientation to avoid the UAVs to flip over while executing desired maneuvers. Furthermore, it is fundamental for both autonomous and remotely controlled UAVs, since even in the latter, the attitude is automatically stabilized through an onboard controller.

This proposed control strategy, has the same problem formulation from the previous one, with the objective to design a feedback controller that manipulates the inputs \((T, \tau_\phi, \tau_\theta, \tau_\psi)\) in order to follow the desired trajectories \(\xi_r(t) = [x_r(t) \ y_r(t) \ z_r(t)]^T\) and \(\psi_r(t)\), and to stabilize the roll and pitch angles. The controller begins with the same two steps of the backstepping design in 4.1, given the system representation as

\[
\dot{\xi}_1 = -k_1 \xi_1 + z_1, \\
\dot{X}_\xi = z_1, \\
\dot{z}_1 = -E_1 - k_2 z_1 - k_x \xi X_\xi + \frac{1}{m} [Re_3 T - \phi_2].
\] (4.65)

In this control strategy it will be considered the presence of external disturbances in the formulation, as in the other of the previous section. For the system (4.65), the change of variable in the previous control strategy on 4.1, was defined as \(z_2 = [Re_3 T - \phi_2]\) or \(z_2 = [Re_3 T - (Re_3 T)_d]\). \((Re_3 T)_d\) is considered the control vector for the translational movements. By evaluating the Euclidean norm of \((Re_3 T)_d\), taking into account that \(\|Re_3\|_2 = 1\), it is verified that \(\|(Re_3 T)_d\|_2 = T_d\) (Hamel et al., 2002; Guenard et al., 2005; Raptis et al., 2011). Thereby, computing the Euclidean norm of \(\|Re_3\|_2 = 1\) yields to

\[
T_d = \|m(ge_3 + \ddot{\xi}_r - k_1 (-k_1 \xi_1 + z_1) - \xi_1 - k_2 z_1 - k_x \xi X_\xi)\|_2,
\] (4.66)

which is the necessary total thrust for the quadrotor UAV to achieve the translational position. Therefore, the desired vector for the translational movements control is given by

\[
Re_3 = \frac{m(ge_3 + \ddot{\xi}_r - k_1 (-k_1 \xi_1 + z_1) - \xi_1 - k_2 z_1 - k_x \xi X_\xi)}{\|m(ge_3 + \ddot{\xi}_r - k_1 (-k_1 \xi_1 + z_1) - \xi_1 - k_2 z_1 - k_x \xi X_\xi)\|_2}.
\] (4.67)
The desired rotation matrix \( R_d \) is obtained by computing the desired angles \( \phi_d, \theta_d \) and \( \psi_d \). The desired yaw angle is defined as the reference \( \psi_d = \psi_r \), and with this constraint, desired roll and pitch angles are obtained by analyzing the third column of the rotation matrix (3.10):

\[
R_{e3} = \begin{bmatrix} C\psi S\theta C\phi + S\psi S\phi \\ S\psi S\theta C\phi - C\psi S\phi \\ C\theta C\phi \end{bmatrix},
\]

and following trigonometric algebra from (4.67) and (4.68), as given by

\[
\begin{align*}
\theta_d &= \text{atan}2(R_d e_3(1) C\psi_d + R_d e_3(2) S\psi_d, R_d e_3(3)) , \\
\phi_d &= \text{atan}2(C\theta_d (S\psi_d R_d e_3(1) - C\psi_d R_d e_3(2)) , R_d e_3(3)),
\end{align*}
\]

where \( \text{atan}2 \) is the arctangent function with in four quadrants, and \( R_d e_3(i) \) is the \( i \) element of the vector \( R_d e_3 \) in (4.67).

- **Third Step**

On related works in the literature, the problem to track the orientation vector of (4.67) is solved via quaternion algebra and Rodrigues’ formula (Hamel et al., 2002; Guenard et al., 2005), or separating the orientation dynamics, including singularity points in the control design (Raptis et al., 2011). In the present thesis, it is proposed to solve this problem by computing a desired rotation matrix to be tracked.

From this step, an orientation controller is designed. Recovering the results from the Second Step in the previous section, given in (4.65), the dynamic of \( \dot{z}_1 \), by adding and subtracting \( \frac{1}{m} R_{e3} T_d \), can be rewritten as

\[
\begin{align*}
\dot{z}_1 &= -\mathcal{E}_1 - k_x z_1 - k_{x\xi} \mathcal{X}_\xi + \frac{1}{m} [R_{e3} T - (R_d e_3 T_d)] + \left[ \frac{1}{m} R_{e3} T_d - \frac{1}{m} R_{e3} T_d \right], \\
&= -\mathcal{E}_1 - k_x z_1 - k_{x\xi} \mathcal{X}_\xi + \frac{1}{m} R_{e3} (T - T_d) + \frac{T_d}{m} (R - R_d) e_3, \\
&= -\mathcal{E}_1 - k_x z_1 - k_{x\xi} \mathcal{X}_\xi + \frac{1}{m} R_{e3} (T - T_d) + \frac{T_d}{m} (R T_d^T - 1_{3x3}) R_d e_3, \\
\end{align*}
\]

where it is defined the error of the total thrust as

\[
\mathcal{E}_T = T - T_d,
\]

and the error of the quadrotor UAV orientation as

\[
\mathcal{E}_R = RR_d T - 1_{3x3},
\]
thus, at this point of the controller design, the control input total thrust is defined as

\[ T = T_d. \]  (4.74)

Asymptotically stability of the system (4.65) is given by (4.14) and Theorem 2. According to Theorem 3.2 in Guenard et al. (2005), asymptotically stability of (4.75) is guaranteed for bounded total thrust \( T = T_d \) and if it is guaranteed the convergence of \( RR_d^T - 1_{3\times3} \). Therefore, the quadrotor dynamic system is rewritten by applying definitions (4.72) and (4.73) in (4.65), and including the orientation error dynamics, which yields

\[
\begin{align*}
\dot{E}_1 &= -k_1E_1 + z_1, \\
\dot{X}_\xi &= z_1, \\
\dot{z}_1 &= -E_1 - k_2z_1 - k_{X\xi} X_\xi + \frac{1}{m} Re_3 E_T + \frac{T_d}{m} E_3 e_3, \\
\dot{E}_R &= RS(\omega) R_d^T - RS(\omega_d) R_d^T, \\
\end{align*}
\]  (4.75)

And the following candidate Lyapunov function is chosen to evaluate the convergence of \( RR_d^T - 1_{3\times3} \)

\[
V_3(E_R) = \frac{1}{2} tr(E_R^T E_R),
\]  (4.76)

where \( tr \) is the trace operator for quadratic matrices. The time derivative of the candidate Lyapunov function (4.76) is given by

\[
\begin{align*}
\dot{V}_3(E_R) &= tr(E_R^T \dot{E}_R), \\
&= tr(E_R^T (RS(\omega) R_d^T - RS(\omega_d) R_d^T)), \\
&= tr ((RR_d^T - I)^T (RS(\omega) R_d^T - RS(\omega_d) R_d^T)), \\
&= tr ((R_d R_d^T - I)(RS(\omega) R_d^T - RS(\omega_d) R_d^T)), \\
&= tr ((R_d R_d^T - I) RS(\omega) R_d^T - RS(\omega_d) R_d^T - I RS(\omega) R_d^T + I RS(\omega_d) R_d^T)) , \\
&= tr (R_d S(\omega) R_d^T - R_d S(\omega_d) R_d^T - RS(\omega) R_d^T + RS(\omega_d) R_d^T) , \\
&= tr (S(R_d \omega) - S(R_d \omega_d) - RS(\omega) R_d^T + RS(\omega_d) R_d^T) , \\
&= tr (S(R_d \omega)) - tr (S(R_d \omega_d)) - tr (RS(\omega) R_d^T) + tr (RS(\omega_d) R_d^T) . \\
\end{align*}
\]  (4.77)

Using the property that trace operator of a skew-symmetric matrix is null \( tr(S(\cdot)) = 0 \), (4.77) becomes

\[
\begin{align*}
\dot{V}_3(E_R) &= - tr (RS(\omega - \omega_d) R_d^T) .
\end{align*}
\]  (4.78)
The virtual control input in this step is chosen as

\[ \phi_3 = \omega_d + k_3 f_1(R, R_d), \]  

(4.79)

where \( k_3 > 0 \) is a constant positive diagonal matrix and the function \( f_1(\xi, \eta, \omega) \) is given by

\[ f_1(R, R_d) = vex(R^T R_d - (R^T R_d)^T). \]

(4.80)

The function \( vex \) is the vector operator that returns the vector from skew-symmetric matrix, and in this case, the chosen term \( (R^T R_d - (R^T R_d)^T) \) is a skew-symmetric matrix. Applying the virtual control input \( \phi_3 \) in the time derivative Lyapunov function (4.78), it is obtained

\[ \dot{V}_3(\mathcal{E}_R) = -tr(RS(\omega_d + k_3 f_1 - \omega_d)R_d^T) \]

\[ = -tr(RS(k_3 f_1)R_d^T) \]

\[ = -tr(R(k_3(R^T R_d - R_d^T R))R_d^T) \]

\[ = -tr(k_3(I - RR_d^T RR_d^T)) \]

\[ = -tr((I - RR_d^T)k_3(I + RR_d^T)) \]

\[ = -\left(k_{3(1,1)}(e_3^T(R^T R_d - R_d^T R)e_2)^2 + k_{3(2,2)}(e_3^T(R^T R_d - R_d^T R)e_3)^2 \right) \]

\[ + k_{3(3,3)}(e_3^T(R^T R_d - R_d^T R)e_3)^2 \]

\[ = -(vex(R^T R_d - R_d^T R)^T k_3 vex(R^T R_d - R_d^T R)) < 0, \]  

(4.81)

which is negative definite.

Adding and subtracting this virtual control law, as \( (RS(\phi_3 - \phi_3)R_d^T) = (RS(\omega_d + k_3 f_1)R_d^T - RS(\omega_d + k_3 f_1)R_d^T) \) into dynamic of \( \mathcal{E}_R \) it is obtained

\[ \dot{\mathcal{E}}_R = RS(\omega)R_d^T - RS(\omega_d)R_d^T + RS(\phi_3 - \phi_3)R_d^T \]

\[ \dot{\mathcal{E}}_n = RS(\omega)R_d^T - RS(\omega_d)R_d^T + RS(\omega_d + k_3 f_1)R_d^T - RS(\phi_3)R_d^T \]

\[ \dot{\mathcal{E}}_n = RS(k_3(vex(R^T R_d - R_d^T R)))R_d^T + RS(\omega - \phi_3)R_d^T \]

\[ \dot{\mathcal{E}}_n = (I - RR_d^T)k_3(I + RR_d^T) + RS(\omega - \phi_3)R_d^T \]

\[ \dot{\mathcal{E}}_n = -\mathcal{E}_n k_3(\mathcal{E}_n + 2I) + RS(\omega - \phi_3)R_d^T \]

\[ \dot{\mathcal{E}}_n = -k_3 \mathcal{E}_n^2 - 2k_3 \mathcal{E}_n + RS(\omega - \phi_3)R_d^T, \]  

(4.82)

which is stable if the term \( RS(\omega - \phi_3)R_d^T \) converges to zero. Therefore it is achieved if \( \omega - \phi_3 \) converges to zero. With the change of variables

\[ z_3 = \omega - \phi_3, \]  

(4.83)
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the complete dynamic of the model can be rewritten as

\[ \dot{E}_1 = -k_1 E_1 + z_1, \]
\[ \dot{X}_\xi = z_1, \]
\[ \dot{z}_1 = -E_1 - k_2 z_1 - k_{X\xi} X_\xi + \frac{1}{m} R e_3 E_T - \frac{T_a}{m} E_R R_d e_3, \]
\[ \dot{E}_R = -k_3 E^2 + 2 k_3 E_R + R S(z_3) R_d^T, \]
\[ \dot{z}_3 = \dot{\omega} - \dot{\phi}_3. \]  
\hspace{1cm} (4.84)

In this step a new candidate Lyapunov function is proposed as

\[ V_4 = V_3 + \frac{1}{2} z_3^T z_3, \]  
\hspace{1cm} (4.85)

and its time derivative is given by

\[ \dot{V}_4(E_R, z_3) = tr \left( E_R^T \dot{E}_R \right) + z_3^T \dot{z}_3 \]
\[ = tr \left( E_R^T (R S(k_3 f_1)) R_d^T + R S(z_3) R_d^T \right) + z_3^T (\dot{\omega} - \dot{\omega}_d - \dot{f}_1) \]
\[ = - f_1^T k_3 f_1 - f_1^T z_3 + z_3^T (\dot{\omega} - \dot{\omega}_d - k_3 \dot{f}_1). \]  
\hspace{1cm} (4.86)

In this step, the virtual control law is chose the time derivative of the angular velocities, \( \dot{\omega} \), as

\[ \phi_4 = \dot{\omega}_d + k_3 \dot{f}_1 + f_1 - k_4 z_3, \quad k_4 > 0, \]  
\hspace{1cm} (4.87)

and the time derivative of the candidate Lyapunov function becomes

\[ \dot{V}_4(E_R, z_3) = - f_1^T k_3 f_1 - z_3^T k_4 \dot{z}_3, \]  
\hspace{1cm} (4.88)

which is negative definite and so, UGAS.

Similarly to the previous controller presented in Section 4.1, the final control input is substituted in quadrotor UAV’s dynamic system (3.33). The difference here is virtual control law \( \phi_4 \), applied as

\[ \tau_a = + I \phi_4 + \omega \times I \omega \]  
\hspace{1cm} (4.89)

The control design on this Section, the stability of (4.65) is achieved by the convergence of (4.81), and the stability analysis of the complete control of this controller is obtained by the sum of the Lyapunov candidate function \( V_3 + V_4 \)

\[ V = V_3 + V_4. \]  
\hspace{1cm} (4.90)
Its time derivative will be given by

\[
\dot{V} = - (vex(R^T R_d - R_d^T R)^T k_3 vex(R^T R_d - R_d^T R)) - f_1^T k_3 f 1 - z_3^T k_4 z_3 < 0.
\]  

which guarantee convergence of the dynamics of (4.11).

### 4.2.1 Simulation Results

Simulations were executed to observe and evaluate the performance of this proposed controller. The same eight-shaped path reference trajectory from the previous Section was used in this simulation. The quadrotor UAV’s parameters are the same of Table 4.1, and it was used also the saturation parameters to limit the control input efforts, bounded by values presented in the table 4.2.

The controller was tuned as a nonlinear PID+FF, similarly to the previous one. The gains without integral action were selected as follows

\[
k_1 = \begin{bmatrix}
3.6 & 0 & 0 \\
0 & 3.6 & 0 \\
0 & 0 & 9.6
\end{bmatrix}, \quad k_2 = \begin{bmatrix}
0 & 0.8 & 0 \\
0 & 0 & 0.8
\end{bmatrix},
\]

\[
k_3 = \begin{bmatrix}
8 & 0 & 0 \\
0 & 8 & 0 \\
0 & 0 & 8
\end{bmatrix}, \quad k_4 = \begin{bmatrix}
32 & 0 & 0 \\
0 & 32 & 0 \\
0 & 0 & 32
\end{bmatrix},
\]

\[
k_{\chi \xi} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
The controller gains matrices with integral action were defined by

\[
k_1 = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},
\]

\[
k_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad k_4 = \begin{bmatrix} 80 & 0 & 0 \\ 0 & 160 & 0 \\ 0 & 0 & 248 \end{bmatrix},
\]

\[
k_{x\xi} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix},
\]

As expected in the control design, this controller showed to be able to follow the time-varying trajectory. In Figure 4.7 illustrated the translational trajectories in the 3D view, and in Figure 4.8 the same translational movements are shown, in this the \( x, \ y, \ z \) trajectories are separated by axes, including the yaw angle tracking in this last one.

Simulations showed convergence of the errors related to the trajectories \( x, \ y, \ z \) and \( \psi \), as expected in the control design, considering the absence of disturbance in this case. The errors are presented in Figure 4.9.

The achievement of the desired roll and pitch angles generated by the controller in the Second Step of the backstepping are showed in Figure 4.10, together with the desired orientation angle yaw. As expected, in this controller the roll and pitch angles are also stabilized. Theses angles are part of the control design, where they are computed in the desired rotation matrix to refer to the angles that will make the quadrotor UAV to achieve the longitudinal and translational path.

The control effort were limited by the saturation parameters, nevertheless the controller is able to achieve the convergence to the reference signals, guarantying stability. This control input signals’ are shown in the figure 4.11, and in the figure 4.12 is detached the first three seconds of the simulation, where it can be observed that both controller, with and without integral action presents limitation of the signals, and can be noted also that the integral action makes the control efforts more aggressive.
Figure 4.7: Trajectory Tracking
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Figure 4.8: Quadrotor UAV Trajectories

(a) Standard Backstepping

(b) Backstepping with Integral Action
(a) Standard Backstepping

(b) Backstepping with Integral Action

Figure 4.9: Trajectories Errors
4.2. BACKSTEPPING ATTITUDE CONTROLLER FOR THE QUADROTOR UAV

(a) Standard Backstepping

(b) Backstepping with Integral Action

Figure 4.10: Quadrotor UAV Orientation
Figure 4.11: Control Input Signals
4.2. BACKSTEPPING ATTITUDE CONTROLLER FOR THE QUADROTOR UAV

(a) Standard Backstepping

(b) Backstepping with Integral Action

Figure 4.12: Control Input Signals - Saturation
4.3 Chapter Conclusions

In this chapter two backstepping controllers were proposed aiming to solve the trajectory tracking for the quadrotor UAV.

The first controller, it was a control law with a vector of the control inputs for the translational movements, formed by the total thrust and the roll and pitch torques. The design of this control avoid singularities and reject constant disturbances with asymptotically stability proof. In this same control strategy, the yaw angle control is designed separated with the backstepping approach, also rejecting constant disturbances, and with asymptotically stability proof, however this control presents singularities due to the Euler angles in the control law. Simulations were performed to this controller, comparing results with the integral action, and without it, showing the rejection of parametric uncertainties and also external disturbances.

The second backstepping control strategy is designed with an initial step that provides the total thrust control input and a desired rotation matrix to track the quadrotor UAV’s orientation. In a second step it is designed a controller that gives the three UAV’s torques that conduces the vehicle to the desired orientation. This controller design avoid singularities in the control equations. Simulations were carried out to show its performance to track a trajectory.
Conclusions

This work proposed contributions on nonlinear control of unmanned aerial vehicles, improving techniques to solve the trajectory tracking problem, under the presence of unmodeled dynamics, parameter uncertainties and also constant external disturbances.

Backstepping control techniques are broaden for the trajectory tracking problem of the quadrotor UAV. A first controller is developed to perform time-varying path tracking of translational position and yaw orientation. It is assumed the whole six degrees of freedom dynamics of the quadrotor, avoiding singularities in the translational control. Furthermore, it is used properties of the rotation matrix to design the control inputs of the states without decoupling the translational and the rotational dynamics, by using vector of the rotation matrix is used as part of virtual controls inputs. The asymptotically stability of the closed-loop system was proved with control Lyapunov functions and Matrosov’s Theorem for time-varying trajectory tracking in the presence of constant disturbances. Simulation results were presented illustrating the difference in the steady-state errors’ behavior, and highlighting that the proposed controller with integral action can guide the system error to zero when constant disturbances affect the UAV.

Another backstepping controller is designed using rotation matrix in the attitude stabilization problem, considering the rotational dynamics linked in the translational control coupled with the dynamics of the orientation angle. In this controller, a backstepping approach obtained the desired total thrust as the first control input, followed by a desired rotation matrix necessary to control the UAV’s orientation.

The difference between the control design of the two controllers is that the first one
used part of the rotation matrix as virtual control inputs, this related the translation movements to the control inputs. In the second controller the complete rotation matrix was used to compute the control inputs. This strategy avoid singularities in the control design.

5.1 Future Works

Some problems discussed in this thesis can be continued in future works.

- To complete the simulations, it can be included noise in the sensor signals and also be simulated in different softwares, as Gazebo Simulator and ProVant Simulator Lara et al. (2017).

- The saturation of the control input signals can be included in the control design.

- In the sequence of this research, real experiments will be performed with quadrotor model AscTec Hummingbird, from Ascending Technologies GmbH, available in the laboratory of MACRO research group. The experiments have already been prepared in the laboratory, and will be carried out to execute the trajectories evaluated in the simulation experiments.

- Quadrotor UAV Euler-Lagrange and Newton-Euler formulations present a strict-feedback structure for three of the six degrees of freedom in the vectorial models. Several papers have already presented nonlinear control techniques using simplified split models, where the states are individualized, and four degrees of freedom are controlled with the four control inputs, but occurring loss of generalities from coupled models. Nonregular backstepping applied to underactuated mechanical systems presents an interesting approach to control the controlled variables and stabilizing the uncontrolled ones.


Maza, I., Kondak, K., Bernard, M., and Ollero, A. (2009). Multi-uav cooperation and control for load transportation and deployment. In *Selected papers from the 2nd...*


