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Novel stability criteria for uncertain delayed Cohen–Grossberg neural networks using discretized Lyapunov functional

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ABSTRACT

This paper deals with the stability analysis of delayed uncertain Cohen–Grossberg neural networks (CGNN). The proposed methodology consists in obtaining new robust stability criteria formulated as linear matrix inequalities (LMIs) via the Lyapunov–Krasovskii theory. Particularly one stability criterion is derived from the selection of a parameter-dependent Lyapunov–Krasovskii functional, which allied with the Gu's discretization technique and a simple strategy that decouples the system matrices from the functional matrices, assures a less conservative stability condition. Two computer simulations are presented to support the improved theoretical results.

1. Introduction

Since the seminal work of John Hopfield [1] showing the relation between recurrent autoassociative neural networks and physical systems, this issue has been extensively studied. Variations of the Hopfield's neural network have been proposed in the literature as, for instance, the one proposed by Cohen and Grossberg in [2] more general than the Hopfield's network. The Cohen–Grossberg model has received great interest due to the potential for applications in classification, parallel computing, associative memory, especially in solving some optimization problems. Particularly, as discussed in [3–5], in the hardware implementation of the neural networks, when communication and response of neurons happens, time-delays may occur. Actually, time-delays are known to be a possible source of instability in many real-world systems in engineering, biology, etc. (see, for example, [6] and references therein). Taking this point, the problem of stability of analogical neural networks with time-delays may be considered under two distinct criteria. The first one is the so-called delay-independent stability criterion which does not explicitly include any information about the value of the time-delay: [7–12]. The other one is concerned with the delay-dependent stability criterion in which the value of the time-delay is explicitly taken into the formulation: [13,5,14–19], so this kind of stability condition may be more applicable than delay-independent one. Besides, in hardware implementation of neural networks, as in any physical system, the uncertainties can arise from parameter fluctuation, modeling errors, some ignored factors, external disturbance, etc. Therefore, this paper has special attention to uncertain state delayed Cohen–Grossberg neural networks (CGNN).

In the literature, when considering LMI based approaches to deal with neural network stability analysis, the strategies usually adopted to obtain less conservative delay-dependent conditions make use of some known machinery as: overbounding cross terms, different Lyapunov–Krasoviskii functional selections and Leibniz–Newton formula manipulations. However, most of these approaches may introduce some degree of conservativeness. On the other hand, Gu in [20] proposed an alternative strategy taking as starting point the discretization of the Lyapunov–Krasovskii functional which is very efficient to the stability analysis problem for linear time-delay systems [20,6]. Recently it was formulated in [19] a

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delay-dependent stability criterion based on the "discretized" Lyapunov–Krasovskii functional for uncertain state-delayed Hopfield neural networks using the quadratic stability concept, that is, the Lyapunov functional matrices are fixed. However, notice that in the context of uncertain systems, parameter-dependent Lyapunov functional selection is known to allow extra degree of relaxation to robust stability (see [21] for further discussion). Concerning the artificial neural networks stability analysis this kind of strategy has been studied in [22].

The main contribution of this paper is to extend and improve the results presented in [19], in the context of two fronts. The first one is to consider a more general class of neural networks, that is, the Cohen–Grossberg. The second front is to derive a new stability criterion (less conservative than the one in [19]) in a completely new fashion taking as starting point the selection of a new parameter-dependent "discretized" Lyapunov–Krasovskii functional combined with an alternative strategy that introduces slack matrices and decouples the Lyapunov functional matrices from the system matrices. The effectiveness of the proposed method is illustrated by two numerical examples.

The following notation is used in this paper: "*" denotes the symmetric terms in a given matrix, the superscript "T" represents the transpose, $M > 0$ (<0) means that the matrix M is positive (negative) definite, diag{·} denotes a diagonal matrix and col{·} denotes a column vector. Use I to stand for the identity matrix of appropriate dimension.

2. Preliminaries

Consider the following delayed Cohen–Grossberg neural network:

$$\frac{\mathrm{d}u_q(t)}{\mathrm{d}t} = -\eta_q[u_q(t)]\left\{b_q[u_q(t)] - \sum_{p=1}^{n} w^0_{pq} g_p[u_p(t)] - \sum_{p=1}^{n} w^1_{pq} g_p[u_p(t-\tau)] + c_q\right\}, \tag{1}$$

where $q = 1,\dots,n, u_q(t)$ is the qth neuron state; $\eta_q(\cdot)$ is the amplification function; $b_q(\cdot)$ denotes the behaved function; $g_q(\cdot)$ denotes the neuron activation function with $g(0) = 0$; w^0_{pq} and w^1_{pq} are the connection weights; and c_q is a constant. Also, the initial condition for (1) is given as $u_q(t) = \varphi_q(t)$, $\forall t \in [-\tau, 0]$.

In this paper the function $\eta_q(\cdot)$ is assumed to be positive and bounded. It is also considered that the functions $b_q(\cdot)$ and $g_q(\cdot)$ are continuous, differentiable, monotonically increasing and bounded.

Henceforward the time index (t) can be omitted when no confusion may arise. Notice that, the system (1) can be modified to take into account the fixed point at the origin, just applying a shift in the fixed point u^*, i.e., $x = u - u^*$. Therefore, the system (1) becomes

$$\frac{\mathrm{d}x_q(t)}{\mathrm{d}t} = -\gamma_q[x_q(t)]\left\{a_q[x_q(t)] - \sum_{p=1}^{n} w^0_{pq} f_p[x_p(t)] - \sum_{p=1}^{n} w^1_{pq} f_p[x_p(t-\tau)]\right\}, \tag{2}$$

where $\gamma_q(x_q) = \eta_q(x_q + u^*_q) - \eta_q(u^*_q)$, $a_q(x_q) = b_q(x_q + u^*_q) - b_q(u^*_q)$ and $f_p(x_p) = g_p(x_p + u^*_p) - g_p(u^*_p)$, for $q, p = 1, 2, \dots, n$. However, in this paper, for simplicity, it is considered the following matrix notation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \Gamma(x)\{-A[x(t)] + W_0(\sigma)f[x(t)] + W_1(\sigma)f[x(t-\tau)]\}, \tag{3}$$

where $x = [x_1, x_2, \dots, x_n]^{\mathrm{T}}$, $f(x) = [f_1(x_1), f_2(x_2), \dots, f_n(x_n)]^{\mathrm{T}}$, $\Gamma(x) = \mathrm{diag}\,[\gamma_1(x_1), \gamma_2(x_2), \dots, \gamma_n(x_n)]$, $A(\sigma) = \mathrm{diag}\,[a_1(x_1), a_2(x_2), \dots, a_n(x_n)]$, $W_0(\sigma) = (w^0_{pq})_{n\times n}$ and $W_1(\sigma) = (w^1_{pq})_{n\times n}$, for $p, q = 1, \dots, n$. The initial conditions for the transformed system is $x(t) = \phi(t)$, $\forall t \in [-\tau, 0]$, where, $\phi = [\phi_1, \phi_2, \dots, \phi_n]^{\mathrm{T}}$ with $\phi_q(x_q) = \varphi_q(x_q + u^*_q) - \varphi_q(u^*_q)$ for $q = 1, 2, \dots, n$. Besides, in the matrices $W_0(\sigma)$ and $W_1(\sigma)$ the σ is used to represent a time-invariant uncertain and is assumed that these system matrices belonging to a polytopic set, $\mathscr{P}$, described by κ vertices, described as

$$\mathscr{P} = \left\{\mathscr{M}(\sigma) : \mathscr{M}(\sigma) = \sum_{v=1}^{\kappa} \sigma_v \mathscr{M}_v; \sigma_v \geqslant 0, \sum_{v=1}^{\kappa} \sigma_v = 1\right\} \tag{4}$$

with $\mathscr{M}(\sigma) = [W_0(\sigma)\ W_1(\sigma)]$.

For the transformed system with the matrix notation in (3), the following assumptions are made.

Assumptions. (H): For any $x(t) \in \mathbb{R}^n$ and $q = 1, 2, \dots, n$ the following hold:

(1) The amplification function $\Gamma[x(t)]$ is bounded and $0 < \Gamma[x(t)] \leqslant \bar{\Gamma}$, where $\bar{\Gamma} = \mathrm{diag}\,(\bar{\gamma}_1, \bar{\gamma}_2, \dots, \bar{\gamma})$, with $\bar{\gamma}_q \in \mathbb{R}, 0 < \bar{\gamma}_q < \infty$;
(2) The behaved function $A[x(t)]$ is bounded and $\frac{A[x(t)]}{x(t)} \geqslant \bar{A}$, where $\bar{A} = \mathrm{diag}\,(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$, with $\bar{a}_q \in \mathbb{R}, 0 < \bar{a}_q < \infty$;
(3) The activation function $f[x(\cdot)]$ is bounded and satisfies, $0 \leqslant \frac{f[x(\cdot)]}{x(\cdot)} \leqslant \bar{F}$, where $\bar{F} = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n)^{\mathrm{T}}$, with $\bar{f}_p \in \mathbb{R}, 0 < \bar{f}_p < \infty$.

Henceforth the class of artificial neural network (3) is considered. Notice that setting $\Gamma(x) = I$ and $A[x(t)] = Ax(t)$ in (3) the CGNN reduces to a Hopfield neural network. Therefore, the results obtained in this paper can be reduced to the stability of a Hopfield neural network, as a particular case.

In order to obtain the main results of this paper, the following parameter-dependent Lyapunov–Krasovskii functional is used:

$$V(x_t,\sigma)=x^{\mathrm{T}}(t)P(\sigma)x(t)+2x^{\mathrm{T}}(t)\int_{-\tau}^{0}Q(\xi,\sigma)x(t+\xi)\mathrm{d}\xi+\int_{-\tau}^{0}\int_{-\tau}^{0}x^{\mathrm{T}}(t+s)R(s,\xi,\sigma)\mathrm{d}sx(t+\xi)\mathrm{d}\xi$$
$$+\int_{-\tau}^{0}x^{\mathrm{T}}(t+\xi)S(\xi,\sigma)x(t+\xi)\mathrm{d}\xi, \tag{5}$$

where $P(\sigma),Q(\xi,\sigma),S(\xi,\sigma)$ and $R(s,\xi,\sigma)$ are parameter-dependent Lyapunov matrix functions.

Note that, it is not an easy task to consider (5), since it can lead a very complex result, which could not be easily computable. Initially, in order to overcome this, is chosen the particular form to the Lyapunov matrices in respect to σ:

$$[P(\sigma),Q(\xi,\sigma),S(\xi,\sigma),R(s,\xi,\sigma)]=\sum_{v=1}^{\kappa}\sigma_v[P^v,Q^v(\xi),S^v(\xi),R^v(s,\xi)] \tag{6}$$

Moreover, it is considered the Gu discretization technique [20]. This technique consists in dividing the delay interval $[-\tau,0]$ in (5) into N segments $[\theta_i,\theta_{i-1}], i=1,\dots,N$ of equal length $h=\tau/N$, where $\theta_i=-ih$. Besides, the continuous matrix functions $Q^v(\xi),S^v(\xi)$ and $R^v(s,\xi)$ are chosen as piecewise linear, then they can be expressed in terms of their values at the dividing points using interpolation formula, i.e., $Q^v(\theta_i+\alpha h)=(1-\alpha)Q_i^v+\alpha Q_{i-1}^v$, $S^v(\theta_i+\alpha h)=(1-\alpha)S_i^v+\alpha S_{i-1}^v$ and

$$R^v(\theta_i+\alpha h,\theta_j+\beta h)=\begin{cases}(1-\alpha)R_{i,j}^v+\beta R_{i-1,j-1}^v+(\alpha-\beta)R_{i-1,j}^v, & \alpha\geqslant\beta\\ (1-\beta)R_{i,j}^v+\alpha R_{i-1,j-1}^v+(\beta-\alpha)R_{i,j-1}^v, & \alpha<\beta\end{cases}$$

for $0\leqslant\alpha\leqslant1, 0\leqslant\beta\leqslant1$ and $i,j=1,\dots,N$. Notice that this functional is completely determined by P^v, Q_i^v,S_i^v and $R_{i,j}^v$.

It is known that the neural network in (3) is asymptotically stable if for a sufficiently small $\epsilon>0$, the Lyapunov–Krasovskii functional and its time-derivative satisfy, respectively, the following conditions $V(x_t,\sigma)\geqslant\epsilon\|x(t)\|^2$ and $\dot V(x_t,\sigma)\leqslant-\epsilon\|x(t)\|^2$.

Then, with the above considerations the next section presents improved conditions to stability analysis of uncertain delayed Cohen–Grossberg neural networks.

3. Main results

In this section, asymptotic stability criterions for a class of Cohen–Grossberg neural networks with time-delay based on LMIs are presented. The issues presented in the previews section will be used along the proof of the main Theorem stated in the sequel.

Theorem 1. *Consider the class of neural networks in (3) where the system matrices are time-invariant and belong to a polytopic set, $\mathscr{P}$, described by κ vertices as in (4). Let $\tau>0$ be a given scalar for the size of the time-delay, and suppose that assumptions in (H) hold. The system (3) is robustly stable if there exist $n\times n$ matrices: $X_1,X_2,P^v=P^{v,T}$, $S_i^v=S_i^{v,T},Q_i^v,R_{i,j}^v=R_{j,i}^{v,T}$, $i,j=0,1,\dots,N,\forall v=1,\dots,\kappa$, such as the following LMIs are satisfied:*

$$\begin{bmatrix}P^v & \tilde Q^v\\ * & \tilde R^v+\tilde S^v\end{bmatrix}>0 \tag{7}$$

and

$$\begin{bmatrix}\Xi^v & D^{s,v} & D^{a,v}\\ * & -R_d^v-S_d^v & 0\\ * & * & -3S_d^v\end{bmatrix}<0 \tag{8}$$

where

$$\tilde Q^v=[Q_0^v\ Q_1^v\cdots Q_N^v],\quad \tilde S^v=\mathrm{diag}\left\{\frac{1}{h}S_0^v\frac{1}{h}S_1^v\cdots\frac{1}{h}S_N^v\right\},\quad \tilde R^v=\left[R_{ij}^v\right],0\leqslant i,j\leqslant N, \tag{9}$$

with the block of matrices R_{ij}^v at positions (i,j) of matrix $\tilde R^v$

$$\Xi^v=\begin{bmatrix}\Xi_{11}^v & -X_1^{\mathrm{T}}+(-\overline{A}+W_0^v\overline{F})^{\mathrm{T}}\overline{\Gamma}X_2 & X_1^{\mathrm{T}}\overline{\Gamma}W_1^v\overline{F}-Q_N^v\\ * & -X_2^{\mathrm{T}}-X_2 & X_2^{\mathrm{T}}\overline{\Gamma}W_1^v\overline{F}\\ * & * & -S_N^v\end{bmatrix} \tag{10}$$

with $\Xi_{11}=X_1^{\mathrm{T}}\overline{\Gamma}(-\overline{A}+W_0^v\overline{F})+(-\overline{A}+W_0^v\overline{F})^{\mathrm{T}}\overline{\Gamma}X_1+Q_0^v+Q_0^{T,v}+S_0^v$,

$$D^{s,v}=[D_1^{s,v}\ D_2^{s,v}\cdots D_N^{s,v}],\quad D^{a,v}=[D_1^{a,v}\ D_2^{a,v}\cdots D_N^{a,v}], \tag{11}$$

and in $D^{s,v}$ and $D^{a,v}$ it follows:

$$D_i^{s,v} = \begin{bmatrix} \frac{h}{2}(R_{0,i-1}^v + R_{0,i}^v) - (Q_{i-1}^v - Q_i^v) \\ \frac{h}{2}(Q_{i-1}^v + Q_i^v) \\ -\frac{h}{2}(R_{i-1,N}^{T,v} + R_{i,N}^{T,v}) \end{bmatrix}, \quad D_i^{a,v} = \begin{bmatrix} -\frac{h}{2}(R_{0,i-1}^v - R_{0,i}^v) \\ -\frac{h}{2}(Q_{i-1}^v - Q_i^v) \\ \frac{h}{2}(R_{i-1,N}^{T,v} - R_{i,N}^{T,v}) \end{bmatrix},$$

$$S_d^v = \mathrm{diag}\{ S_0^v - S_1^v, \quad S_1^v - S_2^v, \quad \ldots, \quad S_{N-1}^v - S_N^v \}, \tag{12}$$

$$R_d^v = \left[R_{di,j}^v\right], \quad 1 \leqslant, i, j \leqslant N, \text{with} \begin{cases} R_{di,j}^v = h(R_{i-1,j-1}^v - R_{i,j}^v), \\ i,j = 1, \ldots, N. \end{cases} \tag{13}$$

Moreover, $\bar{\Gamma}, \bar{A}$ and $\bar{F}$ are bounds given in the assumptions (**H**).

Proof. Consider the Lyapunov–Krasovskii functional selected as in (5). Firstly, this Lyapunov functional satisfies the condition $V(x_t, \sigma) \geqslant \epsilon \|x(t)\|^2$, $\epsilon > 0$ if $S_0^v > S_1^v > \cdots > S_N^v > 0$ and the LMI (7) is satisfied, for details see [6, p.185]. Moreover, if $S_N^v > 0$, the constrain (8) implies that $S_0^v > S_1^v > \cdots > S_N^v > 0$ [6, Prop. 5.22].

From now on, to simplify the notation, the dependence on σ in the Lyapunov and system matrices can be dropped. Differentiating the Lyapunov–Krasovskii functional in (5), along the trajectories of the system in (3), it follows that:

$$\begin{aligned} \dot{V}(x_t) = {} & 2\dot{x}^{\mathrm{T}}(t)\left[Px(t) + \int_{-\tau}^{0} Q(\xi)x(t+\xi)\mathrm{d}\xi\right] + 2\int_{-\tau}^{0} x^{\mathrm{T}}(t+\xi)S(\xi)\dot{x}(t+\xi)\mathrm{d}\xi + 2x^{\mathrm{T}}(t)\int_{-\tau}^{0} Q(\xi)\dot{x}(t+\xi)\mathrm{d}\xi \\ & + 2\int_{-\tau}^{0}\int_{-\tau}^{0} \dot{x}^{\mathrm{T}}(t+s)R(s,\xi)\mathrm{d}s x(t+\xi)\mathrm{d}\xi. \end{aligned} \tag{14}$$

Now considering the system (3), the following null term can be obtained:

$$0 = 2[X_1 x(t) + X_2 \dot{x}(t)]^{\mathrm{T}} \times \{-\dot{x}(t) - \Gamma[x(t)]A[x(t)] + \Gamma[x(t)]W_0 f[x(t)] + \Gamma[x(t)]W_1 f[x(t-\tau)]\}, \tag{15}$$

where the matrices X_1 and X_2 are slack matrices of appropriate dimensions. With the null term (15) and considering the assumptions in (**H**), the following inequality is obtained:

$$0 \leqslant 2[X_1 x(t) + X_2 \dot{x}(t)]^{\mathrm{T}} \times \{-\dot{x}(t) + \bar{\Gamma}[-\bar{A} + W_0\bar{F}]x(t) + \bar{\Gamma}W_1\bar{F}x(t-\tau)\}. \tag{16}$$

Then, integrating by parts (14) and adding the inequality (16), it yields

$$\begin{aligned} \dot{V}(x_t, \sigma) \leqslant {} & \zeta^{\mathrm{T}}\Xi(\sigma)\zeta - 2x^{\mathrm{T}}(t)\int_{-\tau}^{0} \dot{Q}(\xi,\sigma)x(t+\xi)\mathrm{d}\xi + 2\dot{x}(t)\int_{-\tau}^{0} Q(\xi,\sigma)x(t+\xi)\mathrm{d}\xi + 2x^{\mathrm{T}}(t)\int_{-\tau}^{0} R(0,\xi,\sigma)x(t+\xi)\mathrm{d}\xi \\ & - 2\int_{-\tau}^{0} x^{\mathrm{T}}(t-\tau)R(-\tau,\xi,\sigma)x(t+\xi)\mathrm{d}\xi - \int_{-\tau}^{0} x^{\mathrm{T}}(t+\xi)\dot{S}(\xi,\sigma)x(t+\xi)\mathrm{d}\xi \\ & - \int_{-\tau}^{0}\int_{-\tau}^{0} x^{\mathrm{T}}(t+\xi)\left(\frac{\partial R(\xi,s,\sigma)}{\partial \xi} + \frac{\partial R(\xi,s,\sigma)}{\partial s}\right)x(t+s)\mathrm{d}s\mathrm{d}\xi, \end{aligned} \tag{17}$$

where $\zeta = [x(t)\ \dot{x}(t)\ x(t-\tau)]^{\mathrm{T}}$ and $\Xi(\sigma) = \sum_{v=1}^{\kappa}\sigma_v \Xi^v$, with Ξ^v in (10).

Then, following the same lines as in [6, Sec. 5.7], (17) can be rewritten as

$$\begin{aligned} \dot{V}(x_t, \sigma) = {} & \zeta^{\mathrm{T}}\Xi(\sigma)\zeta + 2\zeta^{\mathrm{T}}\int_0^1 \left[D^s(\sigma) + (1-2\alpha)D^a(\sigma)\right]\phi(\alpha)\mathrm{d}\alpha - \int_0^1 \phi^{\mathrm{T}}(\alpha)S_d(\sigma)\phi(\alpha)\mathrm{d}\alpha \\ & - \int_0^1\left[\int_0^1 \phi^{\mathrm{T}}(\alpha)R_d(\sigma)\phi(\beta)\mathrm{d}\alpha\right]\mathrm{d}\beta, \end{aligned} \tag{18}$$

where $D^s(\sigma) = \sum_{v=1}^{\kappa}\sigma_v D^{s,v}$, $D^a(\sigma) = \sum_{v=1}^{\kappa}\sigma_v D^{a,v}$, with $D^{s,v}$ and $D^{a,v}$ given in (11), and $\phi^{\mathrm{T}}(\alpha) = [x^{\mathrm{T}}(t-h+\alpha h)x^{\mathrm{T}}(t-2h+\alpha h)\cdots x^{\mathrm{T}}(t-Nh+\alpha h)]$.

Applying [6, Prop. 5.21] to (18), one can conclude that $\dot{V}(x_t,\sigma) \leqslant -\epsilon\|x(t)\|^2$ $(\epsilon > 0)$ if the LMI (8) is satisfied. Completing the proof. □

Note that, the main idea in the above theorem follows from the identity $\dot{V}(x_t,\sigma) = \sum_{v=1}^{\kappa}\sigma_v\dot{V}^v(x_t)$. Therefore, to verify if $V(x_t,\sigma) \geqslant \epsilon\|x(t)\|^2$ $(\dot{V}(x_t,\sigma) \leqslant -\epsilon\|x(t)\|^2)$ with $\epsilon > 0$ is sufficient to guarantee that $V^v(x_t) \geqslant \epsilon\|x(t)\|^2$ $(\dot{V}^v(x_t) \leqslant -\epsilon\|x(t)\|^2)$ for all $v = 1, \ldots, \kappa$.

Moreover, it is worth to mention that the above theorem assumes that the uncertain system matrices are time-invariant. However, this theorem can be easily adapted to leads with time-varying uncertain as in the following corollary.

Corollary 1. *Consider the class of uncertain neural networks in (3) where the system matrices belong to a polytopic set $\mathscr{M} \in \mathscr{P}$. Let $\tau > 0$ be a given scalar for the size of the time-delay, and suppose that assumptions (**H**) hold. The system (3) is quadratically stable if there exist $n \times n$ matrices: $X_1, X_2, P, S_i = S_i^{\mathrm{T}}, Q_i, R_{ij} = R_{ji}^{\mathrm{T}}, i,j = 0,1,\ldots,N$, satisfying the LMIs (7) and (8), without the superscript v in the Lyapunov matrices: $P^v, Q_i^v, R_{i,j}^v$ and S_i^v.*

The proof of the last corollary is omitted since it follows the same lines of the proof of Theorem 1, using the Lyapunov–Krasovskii functional as in (5), however taking off the dependence in σ.

4. Numerical illustrative examples

To illustrate the results of this paper, two examples are considered. The first example makes computational experiments to verify the efficiency of the proposed methodology considering a precisely known Cohen–Grossberg time-delayed neural network, also the results obtained are compared with other ones in the literature. The second example considers a Cohen–Grossberg time-delayed neural network with uncertain parameters. In the both examples is considered that the neural network is stable when $\tau = 0$.

Example 1. Consider the following Cohen–Grossberg neural network as in (3), with the functions: $\Gamma(x) = \text{diag}\,[\gamma_1 \tanh(x_1), \gamma_2 \tanh(x_2), \gamma_3 \tanh(x_3)]$, $A(x) = \text{diag}\,(a_1x_1, a_2x_2, a_3x_3), f(x) = \text{col}\,[b_1 \tanh(x_1), b_2 \tanh(x_2), b_3 \tanh(x_3)]$ and the following connection matrices, $W_0 = 0$ and

$$W_1 = \begin{bmatrix} -0.1052 & -0.5069 & -0.1121 \\ -0.0257 & -0.2808 & 0.0212 \\ 0.1205 & -0.2153 & 0.1315 \end{bmatrix}.$$

Firstly consider the following bounds in the assumptions (**H**): $\bar{\Gamma} = I, \bar{A} = \text{diag}\,[4.1898, 0.7160, 1.9985]$ and $\bar{F} = \text{diag}\,[0.4219, 3.8993, 1.0160]$. Notice that with this configuration, the methods in [14,15,13,16,17,19] can be applied. The problem to be addressed is to search for the largest time-delay such that the neural network remains stable. Using Theorem 1, with different number of partitions on the time-delay interval, i.e. different value to N, the results are presented in the Table 1.

For comparison purpose, the approaches developed in [14,15,18,13,16,17,19] are also considered for this neural network system. Table 2 presents the largest time-delay obtained by each one (notice that the approaches in [14] and [15] are infeasible). Notice that, by Table 2, the best result has been obtained using the approach in [19], which produces the same result when using the proposed approach, see Table 1. However the approach in [19] cannot be applied to all Cohen–Grossberg neural networks class.

Now considering the upper bound to the amplification function as $\bar{\Gamma} = \bar{\gamma} I$, for $0 < \bar{\gamma} \neq 1 < \infty$, the approach in [19] cannot be applied. Table 3 shows the largest time-delay obtained when applying Theorem 1, for different values of $\bar{\gamma}$.

Example 2. Consider the following Cohen–Grossberg neural network:

$$\begin{cases} \frac{dx_1}{dt} = [1 + \sin(x_1)]\{-1x_1 + (-1+\rho)\tanh(x_2) + (-1+\rho)\tanh[x_2(t-\tau)]\}, \\ \frac{dx_2}{dt} = [2 + \cos(x_2)]\{-1.5x_2 + (1+\rho)\tanh(x_1) + (2+\rho)\tanh[x_1(t-\tau)]\}, \end{cases} \tag{19}$$

where ρ represents a parameter fluctuation or modelling errors. Assume that ρ belongs to the uncertain domain $[-\hat{\rho}, \hat{\rho}]$.

This neural network can be represented as in (3) with the following terms:

$$\Gamma(x) = \begin{pmatrix} 1+\sin(x_1) & 0 \\ 0 & 2+\cos(x_2) \end{pmatrix}, \quad A(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$W_0 = \begin{pmatrix} 0 & -1+\rho \\ 1+\rho & 0 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 0 & -1+\rho \\ 2+\rho & 0 \end{pmatrix},$$

and $f[x(\cdot)] = \tanh[x(\cdot)]$. Then considering the constraints in (**H**), the following bounds are imposed, $\bar{\Gamma} = \text{diag}\,(2,3), \bar{A} = \text{diag}\,(1, 1.5)$ and $\bar{F} = \text{diag}\,(1,1)$.

Initially consider the special case with $\rho = 0$. Applying Theorem 1, the results in Table 4 are obtained.

Now considering the uncertain Cohen–Grossberg neural network described in (19), then the proposed method presented in the Corollary 1 can be applied independent of the variations on the uncertain parameter, i.e. $\rho(t) \in [-\hat{\rho}, \hat{\rho}]$, but the Theorem 1 is preferred to be applied when the uncertain parameter is time-invariant, i.e. $\rho \in [-\hat{\rho}, \hat{\rho}]$. However, will be shown now that the Corollary 1 obtains worst results than the parameter-dependent approach as presented in Theorem 1.

First the problem of obtaining the maximum possible domain of stability, i.e. to seek for the largest $\hat{\rho}$, fixing a time-delay, say $\tau = 0.5$, is investigated. Applying Corollary 1 with $N = 2$ is obtained $\hat{\rho} = 0.2697$, and applying Theorem 1 with $N = 2$ is obtained $\hat{\rho} = 0.4114$. It is easy to note that the parameter-dependent approach allows to find greater bounds to the uncertain domain.

Table 1
The largest time-delays using Theorem 1, with different values to N (Example 1)

N	1	2	3	4	5	6
τ	2.7466	2.7715	2.7729	2.7731	2.7732	2.7732

Table 2
The largest time-delay by the approaches in [14,15,13,16–19] (Example 1)

Method in	[14]	[15]	[13]	[16]	[17]	[18]	[19] (N = 2)
τ	–	–	0.4121	1.7484	1.7644	2.2056	2.7715

Table 3
The largest time-delays obtained using Theorem 1 with $N = 2$, for different values of $\bar{\gamma}$ (Example 1)

$\bar{\gamma}$	2	4	6	8	10
τ	1.3857	0.6928	0.4619	0.3464	0.2771

Table 4
The largest time-delay for $\rho = 0$ using Theorem 1, with different values to N (Example 2)

N	1	2	3	4	5	6	7
τ	0.9597	1.0514	1.0764	1.0783	1.0786	1.0787	1.0787

On the other hand, a different case can be considered: to search the largest time-delay, τ_{max}, in such way that the stability holds for a fixed uncertain domain $\rho \in [-\hat{\rho}, \hat{\rho}]$. If one choose $\hat{\rho} = 0.2$ and applies Corollary 1 with $N = 2$ the maximum time-delay obtained is $\tau_{\max} = 0.7054$. However, applying Theorem 1 with $N = 2$, the maximum time-delay is $\tau_{\max} = 0.9675$. Again notice that the parameter-dependent approach improves the result in terms of guaranteeing stability for a larger time-delay, considering a fixed uncertain domain.

5. Conclusions

The problem of robust stability analysis of uncertain delayed Cohen–Grossberg neural networks has been analyzed. An LMI based approach has been derived through the selection of an appropriate parameter-dependent Lyapunov–Krasovskii functional, utilizing the Gu's discretization technique allied with a simple strategy that decouples the system matrices from the functional ones.

The efficiency of the proposed method, is illustrated by simple examples and compared with other recent methods in the literature, achieving better performance.

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